# Optimization Problems with Equilibrium Constraints

## GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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# Outline

1. Solving MPECs as NLPs
2. Convergence for Sequential Quadratic Programming Methods
3. Convergence for Interior-Point Methods
4. An SLPEC-EQP Approach
   - Counter Example for SQPEC
   - SLPEC Method
   - Accelerating Local Convergence

# Solving MPECs as NLPs

Mathematical Program with Equilibrium Constraints (MPEC)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \perp F(x,y) \geq 0 \end{cases}$$

Equivalent smooth (lazy) nonlinear program (NLP):

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) = s, \quad s \geq 0, \quad y \geq 0 \quad \text{and} \quad y^T s \leq 0 \end{cases}$$

# Switching Notation

To understand convergence analysis, we switch notation:
$x = (x_0, x_1, x_2)$:

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \geq 0 \\ & 0 \leq x_1 \perp x_2 \geq 0 \end{cases}$$

Equivalent smooth nonlinear program (NLP):

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \geq 0 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_1^T x_2 \leq 0 \end{cases}$$

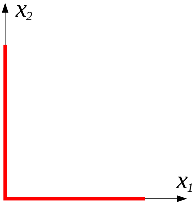
Now examine convergence properties of NLP solvers ...

# A Nonlinear Programming Approach

Replace equilibrium $0 \le x_1 \perp x_2 \ge 0$ by $X_1 x_2 \le 0$ or $x_1^T x_2 \le 0$

$\Rightarrow$ standard nonlinear program (NLP)

$$
(\text{NLP}) \left\{ \begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \ge 0 \\ & x_1, \; x_2 \ge 0 \\ & \boxed{X_1 x_2 \le 0} \end{array} \right.
$$




Advantage: standard (?) NLP; use large-scale solvers ...
Snag: nonlinear program (NLP) violates standard assumptions!

# Strong Stationarity & Unbounded Multipliers

Example $x^* = (0, 1)$:

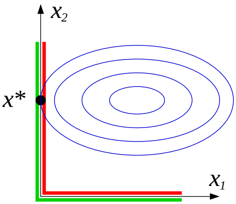
$$\begin{cases} \min\limits_x \frac{1}{2}(x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{s.t. } x_1, x_2 \geq 0,\ x_1 x_2 \leq 0 \end{cases}$$

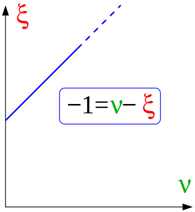
first order conditions:

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \nu_1 \\ 0 \end{pmatrix} - \begin{pmatrix} \xi \\ 0 \end{pmatrix}$$

$\nu_1$ multiplier of $x_1 \geq 0$; $\xi$ multiplier of $x_1 x_2 \leq 0$.

Equivalent NLP ($x_1 x_2 \leq 0$) violates MFCQ $\Rightarrow$ unbounded multipliers




multipliers form a ray $\Rightarrow \exists$ bounded multipliers

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# The Relaxed NLP

Define index sets

$$\mathcal{X}_1 := \{i : x^*_{1i} = 0\} \qquad \& \qquad \mathcal{X}_2 := \{i : x^*_{2i} = 0\},$$

complements $\mathcal{X}_j^\perp := \{1, \ldots, p\} - \mathcal{X}_j$

$\Rightarrow$ relaxed NLP given by

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \geq 0 \\ & x_{1j} = 0 \quad \forall j \in \mathcal{X}_2^\perp \\ & x_{2j} = 0 \quad \forall j \in \mathcal{X}_1^\perp \\ & x_1, x_2 \geq 0 \end{cases}$$

... i.e. $\mu_i$ multiplier of "equality" constraints

# Equivalence to KKT Conditions

KKT conditions of equivalent NLP: $\exists \lambda^*, \nu_1^*, \nu_2^*, \xi^* \geq 0$

$$\nabla f(x^*) - \nabla c(x^*)^T \lambda^* - \begin{pmatrix} 0 \\ \nu_1^* - X_2^* \xi^* \\ \nu_2^* - X_1^* \xi^* \end{pmatrix} = 0 \;\; 1^{st} \text{ order}$$

$$c(x^*) \geq 0, \;\; x_1^* \geq 0, \;\; x_2^* \geq 0 \;\; \text{and} \;\; X_1^* x_2^* \leq 0 \;\; \text{primal feas.}$$

$$c(x^*)^T \lambda \; = \; {x_1^*}^T \nu_1^* \; = \; {x_2^*}^T \nu_2^* = 0 \;\; \text{compl. slack.}$$

... $\xi > 0$ allows $\mu_1 < 0$

... multipliers of relaxed NLP $\mu_1 = \nu_1 - X_2^* \xi$, and $\mu_2 = \nu_2 - X_1^* \xi$
$\Rightarrow$ KKT multipliers bounded if $\|\xi^*\| < \infty$

# Convergence of SQP for MPECs

Sequential Quadratic Programming (SQP) ... compute step $d$

$$\left\{\begin{array}{ll} \min\limits_x & f(x) \\ \text{s.t.} & c(x) \geq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & X_1 x_2 \leq 0 \end{array}\right. \rightarrow \left\{\begin{array}{ll} \min\limits_d & \nabla f_k^T d + \frac{1}{2} d^T H_k d \\ \text{s.t.} & c_k + \nabla c_k^T d \geq 0 \\ & x_{k1} + d_1 \geq 0 \\ & x_{k2} + d_2 \geq 0 \\ & X_{k1} x_{k2} + X_{k1} d_2 + X_{k2} d_1 \leq 0 \end{array}\right.$$

where $H_k \simeq \nabla^2 f_k - \sum \lambda_i \nabla^2 c_k$ Hessian of the Lagrangian.

Set $x_{k+1} = x_k + d$ & update multiplier estimates

# Convergence of SQP for MPECs

Sequential Quadratic Programming (SQP) ... compute step $d$

$$\begin{cases} \min\limits_x f(x) \\ \text{s.t. } \; c(x) \geq 0 \\ \qquad x_1 \geq 0 \\ \qquad x_2 \geq 0 \\ \qquad X_1 x_2 \leq 0 \end{cases} \rightarrow \begin{cases} \min\limits_d \nabla f_k^T d + \frac{1}{2} d^T H_k d \\ \text{s.t. } \; c_k + \nabla c_k^T d \geq 0 \\ \qquad x_{k1} + d_1 \geq 0 \\ \qquad x_{k2} + d_2 \geq 0 \\ \qquad X_{k1} x_{k2} + X_{k1} d_2 + X_{k2} d_1 \leq 0 \end{cases}$$

where $H_k \simeq \nabla^2 f_k - \sum \lambda_i \nabla^2 c_k$ Hessian of the Lagrangian.

Set $x_{k+1} = x_k + d$ & update multiplier estimates

Two cases: $\exists k: \; X_{k1} x_{k2} = 0$ ... or ... $X_{k1} x_{k2} > 0, \; \forall k$

# Convergence of SQP Part 1: $X_{k1} x_{k2} = 0$

wlog have $x_{k1} = 0$ (and for simplicity assume $x_{k2} > 0$)

$\Rightarrow$ QP contains constraints

$$\left.\begin{aligned} x_{k1} + d_1 &\geq 0 \\ x_{k2} + d_2 &\geq 0 \\ X_{k1} x_{k2} + X_{k2} d_1 + X_{k1} d_2 &\leq 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} d_1 &\geq 0 \\ x_{k2} + d_2 &\geq 0 \\ X_{k2} d_1 &\leq 0 \end{aligned}\right\} \Rightarrow d_1 = 0$$

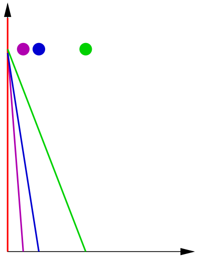
$\Rightarrow x_1^{k+1} = x_{k1} + d_1 = 0$ ... stay on same axis
$\Rightarrow$ same tangent cone as NLP with $x_1 = 0$ ... relaxed NLP
$\Rightarrow$ fast local convergence

# Convergence of SQP Part 2: $X_{k1}x_{k2} > 0$

wlog $x_1^* = 0$, but $X_{k1}x_{k2} > 0$, i.e. off axis

QP picks nonsingular basis, subset of

$$\begin{bmatrix} & 0 & 0 \\ \nabla c_k & I & X_{k2} \\ & 0 & X_{k1} \end{bmatrix}$$



Assume all QPs consistent ... 2 cases:

case 1: true subset $\Rightarrow$ non-singular $\Rightarrow$ quadratic convergence

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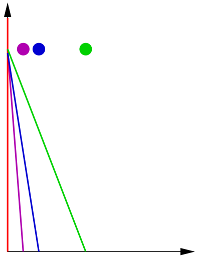


Assume all QPs consistent ... 2 cases:

case 1: true subset $\Rightarrow$ non-singular $\Rightarrow$ quadratic convergence

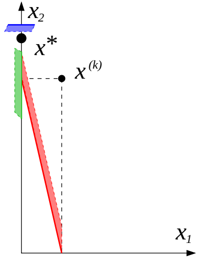
case 2: full set $\Rightarrow$ $x_{k1} > 0$ (otherwise singular)
$\Rightarrow$ $X_1^{k+1}x_2^{k+1} = 0$ now see Part (1) as before ...

# Consistency of QP Approximations

Are QPs always consistent for MPECs?

NO! Linearization can be inconsistent arbitrarily close to solution

$$\begin{cases} \underset{x}{\text{minimize}} & x_1 + x_2 \\ \text{subject to} & x_2^2 \geq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 \; x_2 \leq 0 \end{cases}$$

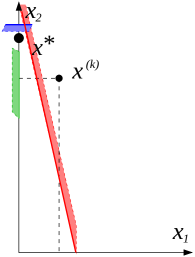



generic problem $\Rightarrow$ solvers take arbitrary steps

# Consistency of QP Approximations

Relax linearization of $X_1 x_2 \leq 0$ ...
... heuristic for infeasible QPs ($0 < \delta, \kappa < 1$ constants)

$$X_{k1} x_{k2} + X_{k2} d_1 + X_{k1} d_2 \leq \delta \left( x_{k1}^T x_{k2} \right)^{1+\kappa} e$$




... works in well practice with $\delta = 0.1, \ \kappa = 1$

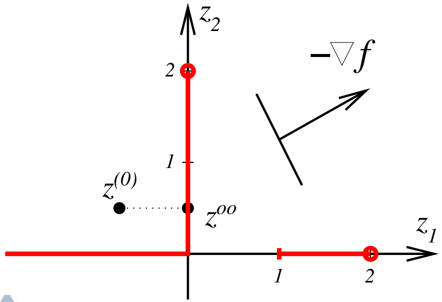
# The Slacks Matter!!!

How important was the introduction of slack variables?

Consider MPEC without slacks ...

$$(P) \begin{cases} \underset{z}{\text{minimize}} & -x_1 - \frac{1}{2}x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 0 \leq x_1^2 - x_1 \perp x_2 \geq 0 \, . \end{cases}$$

with solutions $(2,0)^T$ with $f^* = -2$ and $(0,2)^T$ with $f^* = -1$




- Start $(-\epsilon, t)^T$
- Nonstationary limit $(0, t)^T$ for any $t$.
- Avoid failure with slacks

# Outline

# Interior Point Penalty Methods for MPECs

Equivalent NLP:

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \geq 0 \\ & x_1 \geq 0, \quad x_2 \geq 0, \\ & x_1^T x_2 \leq 0 \end{cases}$$

Consider $\ell_1$ penalty of complementarity constraint

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) + \pi x_1^T x_2 \\ \text{subject to} & c(x) \geq 0 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{cases}$$

... form primal-dual system with a twist ...

# Interior Point Penalty Methods for MPECs

Primal-dual MPEC system with $x_1, x_2$ in primal form

$$\begin{cases} \nabla f(x) - \nabla c(x)^T \lambda - \begin{pmatrix} 0 \\ \mu X_1^{-1} e - X_2 \pi \\ \mu X_2^{-1} e - X_1 \pi \end{pmatrix} = 0 \\ c(x) - s = 0 \\ S\lambda = \mu e \end{cases}$$

Algorithm I: Interior Penalty Method for MPECs

1. Choose barrier parameter $\mu_k$, and tolerance $\epsilon_k$
2. Solve PD system to tolerance $\epsilon_k$ and ensure

$$\| \min\{x_{k1}, x_{k2}\} \| \le \sqrt{\epsilon_k} \qquad \text{by adjusting } \pi_k$$

# Interior Point Penalty Methods for MPECs

**Theorem**

*If Algorithm I generates an infinite sequence, then:*

1. $x_k \to x^*$ *is feasible,*
2. *LICQ for relaxed NLP* $\Rightarrow x^*$ *is C-stationary,*
3. $\pi_k x_{ki} \to 0 \Rightarrow x^*$ *strongly stationary,*
4. *superlinear convergence for suitable barrier updates*

Practical implementation

- dynamic penalty $\pi_k$ update during inner iteration
- non-monotone reduction of complementarity: $\pi^j = 10\pi^j$ if,

$$x_1^{j^T} x_2^j > 0.9 \max\left\{ x_1^{(j-1)^T} x_2^{(j-1)}, \ldots, x_1^{(j-m+1)^T} x_2^{(j-m+1)} \right\}$$

- avoid trouble with badly scaled MPECs

# Relaxed Interior Point Methods for MPECs

Perturb rhs of complementarity constraint ... $X_1 x_2 \leq C\mu e$

... where $\mu > 0$ barrier parameter $\Rightarrow$ primal dual system ...

$$\begin{cases} \nabla f(x) - \nabla c(x)^T \lambda - \begin{pmatrix} 0 \\ \mu X_1^{-1} e - X_2 \xi \\ \mu X_2^{-1} e - X_1 \xi \end{pmatrix} = 0 \\ c(x) - s = 0 \\ S\lambda = \mu e \\ X_1 x_2 + t = C\mu e \\ T\xi = \mu e \end{cases}$$

$\Rightarrow$ central path $(x(\mu), \nu(\mu), \xi(\mu))$ for $\mu > 0$
[Raghunathan and Biegler, 2002, Liu and Sun, 2002]

# Relaxed Interior Point Methods for MPECs

Compare relaxation and penalization

$$
\begin{aligned}
\xi &= \pi \\
t &= C\mu e - X_1 x_2 \\
T\xi &= \mu e
\end{aligned}
$$

$\Rightarrow$ Penalization $\Leftrightarrow$ Relaxation, if

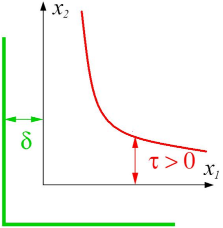
$$\pi_i = \frac{\mu}{\mu C_i - x_{1i} x_{2i}} \quad \text{or} \quad C_i = \frac{\mu + \pi_i x_{1i} x_{2i}}{\mu \pi_i}$$

... convergence proofs carry over!

# Interior Point Method with Two Sided Relaxation

Clever idea by Friedlander, de Miguel & Scholtes [2003]:

- MPECs have no strict interior
- Relax $X_1 x_2 \le \tau e \Rightarrow$ interior $\to 0$
  $\Rightarrow$ relax $X_1 x_2 \le \tau e$
  and $x_1 \ge -\delta e$, $x_2 \ge -\delta e$
- Adjust $\tau, \delta$ as $\mu \to 0$




**Theorem**

*In limit $\tau \to 0$ or $\delta \to 0$ but not both*
*$\Rightarrow$ relaxed problem has non-empty interior in limit*
*$\Rightarrow$ interior point methods faster & more robust*

MPEC multiplier $\mu_i < 0 \Rightarrow$ reduce $\tau_i \searrow 0$ ...

# Outline

# SQPEC Approach [Scholtes, 2004]

Sequential QPEC approach (similar to piecewise SQP)

$$\begin{aligned} \underset{d}{\text{minimize}} \quad & {g^{(k)}}^T d + \tfrac{1}{2} d^T H^{(k)} d \\ \text{subject to } & c^{(k)} + {A^{(k)}}^T d \geq 0, \\ & 0 \leq x_1^{(k)} + d_1 \;\perp\; x_2^{(k)} d_2 \geq 0 \end{aligned}$$

where $g^{(k)} = \nabla f(x^{(k)})$ and $A^{(k)} = \nabla c(x^{(k)}, y^{(k)})$,

Solve sequence of QPECs, set $x^{(k+1)} = x^{(k)} + d$

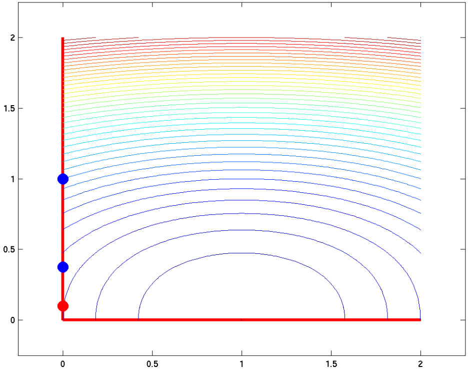
**Theorem [Scholtes, 04]:** Local B-stationary convergence.

SQPEC has correct tangent cone $\Rightarrow$ global convergence???

## **No!** Counter Example for SQPEC

Consider

minimize $(x_1 - 1)^2 + x_2^3 + x_2^2$ subject to $0 \leq x_1 \perp x_2 \geq 0$




SQPEC: $x^{(k+1)} = \left(0, 3{x_2^{(k)}}^2/(6x_2^{(k)} + 2)\right) \to (0,0)$ spurious

# A Sequential LPEC Method

**while** (not optimal) **begin**

1. Compute step $d$ from LPEC subproblem

$$\begin{aligned} \underset{d}{\text{minimize}} \quad & g^{(k)^T} d \\ \text{subject to} \quad & c^{(k)} + A^{(k)^T} d \geq 0, \\ & 0 \leq x_1^{(k)} + d_1 \perp x_2^{(k)} + d_2 \geq 0 \\ & \|d\|_\infty \leq \Delta_k \qquad \text{trust-region} \end{aligned}$$

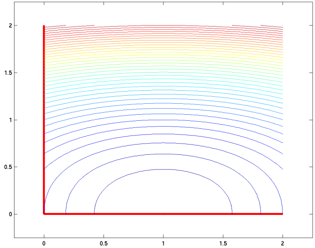
2. **if** $x^{(k)} + d$ acceptable **then**
   $x^{(k+1)} = x^{(k)} + d$ & increase TR $\Delta^{(k+1)} = 2 * \Delta_k$
   **else** $x^{(k+1)} = x^{(k)}$ & decrease TR $\Delta^{(k+1)} = \Delta_k / 2$

**end**

1. Like steepest descend: Can we speed up convergence?
2. When is $x^{(k)} + d$ acceptable?
3. How do we solve the LPEC subproblem?

# Spurious A/M-Stationarity Revisited

Consider min $(x_1 - 1)^2 + x_2^3 + x_2^2$ subject to $0 \leq x_2 \perp x_1 \geq 0$




- SLPEC pivots through $(0, 0)$ ... get onto $x_1$-axis
- SLPEC converges to B-stationary limit $(1, 0)$
- ... cannot get stuck in spurious stationary points

# Spurious A/M-Stationarity Revisited

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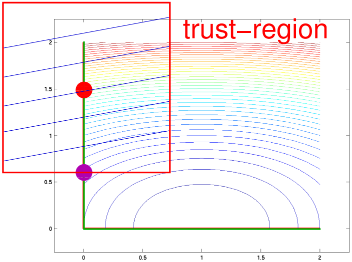


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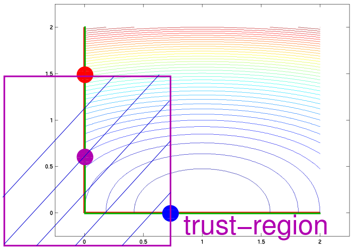


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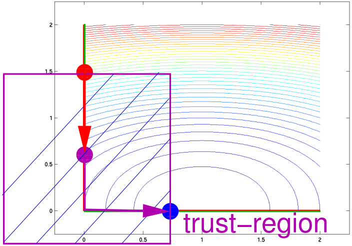


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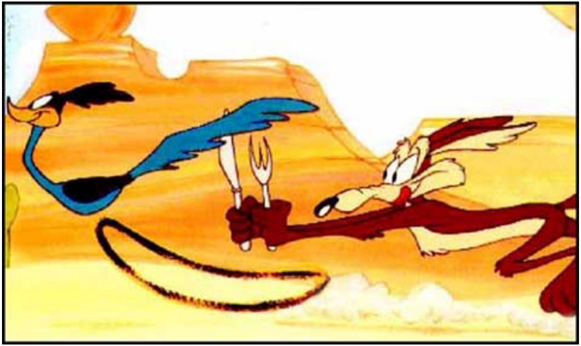
# Accelerating Local Convergence



# Equality Constrained Quadratic Program (EQP)

Given active set estimate from LPEC step $d$:

$$\mathcal{A}_c(d) := \{ i : c_i^{(k)} + a_i^{(k)^T} d = 0 \}$$
$$\mathcal{A}_1(d) := \{ i : x_{1i}^{(k)} + d_{1i} = 0 \}$$
$$\mathcal{A}_2(d) := \{ i : x_{2i}^{(k)} + d_{2i} = 0 \}$$

solve corresponding equality QP

$$\text{EQP}_k(d) \begin{cases} \underset{s}{\text{minimize}} & g^{(k)^T} s + \frac{1}{2} s^T H^{(k)} s \\ \text{subject to} & c_i^{(k)} + a_i^{(k)^T} s = 0, \quad \forall i \in \mathcal{A}_c(d) \\ & x_{1i}^{(k)} + s_{1i} = 0, \quad \forall i \in \mathcal{A}_1(d) \\ & x_{2i}^{(k)} + s_{2i} = 0, \quad \forall i \in \mathcal{A}_2(d) \end{cases}$$

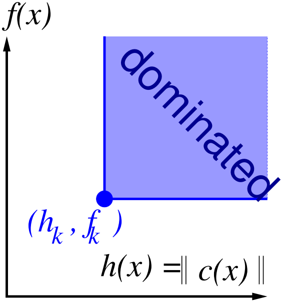
for 2nd order step $s$.

# A Filter Method for MPECs

MPEC have three competing aims

1. Minimize $f(x, y)$
2. Minimize $h(x, y) := \|c^-(x, y)\|$ ... more important
3. Minimize $h^c(x, y) := \|\min(x_1, x_2)\|$ ... most important

... for plots, let $h(x) := h(x, y) + h^c(x, y)$




Borrow concept of domination from multi-objective optimization

$$(h_k, h_k^c, f_k) \text{ dominates } (h_l, h_l^c, f_l)$$
$$\text{iff} \quad h_k \le h_l \ \& \ h_k^c \le h_l^c \ \& \ f_k \le f_l$$

i.e. $(x^{(k)}, y^{(k)})$ at least as good as $(x^{(l)}, y^{(l)})$

# Global Convergence to B-Stationarity

**Assumptions:**

- MPEC-MFCQ (i.e. every piece satisfies MFCQ) weak
- $x^{(k)}$ remain in compact set strong
- $f$, $c$ twice continuously differentiable

**Theorem**

*Outcome of SLPEC is one of:*

1. *restoration phase fails to find feasible point, or*
2. $d = 0$ *solves LPEC* $\Rightarrow$ *B-stationary, or*
3. *limit is B-stationary.*

**Proof:** exploit fact that LPEC $\equiv$ disjunctive LPs

# Conclusions

Considered convergence of four classes of methods for MPECs

1. SLPEC-EQP Method
   - Method of choice, but LPEC hard to solve
   - Developing active-set type solver for LPECs
     $\Rightarrow$ base on standard LP solvers ...
2. Sequential Quadratic Programming Methods
   - Often works very well ... my preferred method
   - Fails to converge or converges slowly for degenerate MPECs
3. Interior-Point Methods
   - Works mostly well ... not as robust as SQP
   - Fails to converge or converges slowly for degenerate MPECs
4. Sequential penalization or regularization methods
   - Not as effective as SQP or IPM above
     ... solve sequence of NLPs versus a single one!
   - Fails to converge or converges slowly for degenerate MPECs

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