# Tutorial 10: Solving Cutting Stock Problem Using Column Generation Technique

## GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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## Linear Program in standard form:

$$\begin{aligned} \text{Minimize } & c^T x, \\ \text{Subject to } & Ax = b, \\ & x \geq 0. \end{aligned}$$

- $\mathcal{P}$ is the corresponding feasible set, matrix $A_{m\times n}$ has full row rank
- $A_j$ is the $j^{th}$ column of the matrix $A$.
- $x$ be a 'basic' feasible solution and $B(1), \ldots, B(m)$ be the indices of basic variables.
- $NB = \{NB(1), \ldots, NB(n-m)\}$ be the indices of nonbasic variables.
- $B = \left[A_{B(1)} \ldots A_{B(m)}\right]$ is basis matrix and $N = \left[A_{NB(1)} \ldots A_{NB(n-m)}\right]$ be the nonbasis matrix.

## Optimality conditions

- Vector $x_N = \left(x_{NB(1)}, \ldots, x_{NB(n-m)}\right)$ for nonbasic variables is $\mathbf{0}$ and vector $x_B = \left(x_{B(1)}, \ldots, x_{B(m)}\right)$ of basic variables is obtained as

$$
\begin{aligned}
Ax = [B \;\; N]^T \begin{bmatrix} x_B \\ x_N \end{bmatrix} &= b, \\
Bx_B + Nx_N &= b, \\
x_B &= B^{-1} b.
\end{aligned}
$$

Consider moving away from $x$ to $x + \theta d^j, \;\theta > 0$ by selecting a nonbasic variable $x_j, \; j \in N$.
**Algebraically**, $d^j_j = 1$ and $d^j_i = 0, \forall i \in NB, i \neq j$.
$x_B$ becomes $x_B + \theta d^j_B$, where $d^j_B = \left(d^j_{B(1)}, \ldots, d^j_{B(m)}\right)$

For feasibility, $A(x + \theta d^j) = b$ and

$$\begin{aligned} Ad^j &= Bd^j_B + Nd^j_N = 0, \\ Bd^j_B + A_j &= 0, \\ d^j_B &= -B^{-1}A_j. \end{aligned}$$

Objective value at the new point is $c^T(x + \theta d^j)$ and per unit change along basic direction $d^j$ (reduced cost of nonbasic variable $x_j$) is

$$\bar{c}_j = c^T d^j = c_j - c_B^T B^{-1} A_j$$

**Theorem (Optimality conditions)**

*Consider a basic feasible solution $x$ associated with a matrix $B$, and let $\bar{c}$ be the corresponding vector of reduced costs. If $\bar{c} \geq 0$, then $x$ is optimal.*

# Column Generation

Motivation:

- Column generation first suggested in the context of multi-commodity network flow problem (Ford and Fulkerson, 1958).
- Dantzig and Wolfe (1960) adapted it to LP with a decomposable structure.
- Gilmore and Gomory (1961) demonstrated its effectiveness in a cutting stock problem.
- Other applications: Vehicle routing, crew scheduling, integer-constrained problems etc.

# Column Generation

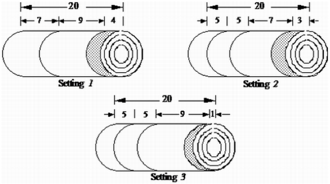
Recall

- number of nonzero variables (basic variables) is equal to the number of constraints.
- Hence even though the number of possible variables (columns) may be large, we only need a small subset of these (in basis B) in the optimal solution.

Crucial insight

- If a problem has many variables (or columns) but fewer constraints, work with a partial $A$ matrix.

# Example: Cutting Stock Problem




| | | | |
|---|---|---|---|
| Size of the item demanded | 5-ft | 7-ft | 9-ft |
| Number of items demanded | 25 | 20 | 15 |

- Width of standard stock is 20 feet.
- Demand is met by cutting up standard stocks into items of required widths (refer figure).
- Objective is to minimize the number of standard stocks to meet the customer demands.

# Cutting Stock Problem (CSP)

Problem description:

- Stock width $W_S$, and a set of items $\mathcal{I}$.
- Width of items denotd by $w_i$, and their demand $d_i$.
- Cost of using a stock per unit width is 1
- Set of cutting patterns $\mathcal{P}$
- $a_{ip}$: number of pieces of item $i \in \mathcal{I}$ cut in pattern $p \in \mathcal{P}$
- Minimize total cost (number of stocks used)

Decision variables:

- $x_p \in \mathcal{P}$: number of times a cutting pattern $p$ is used

# Mathematical formulation (master)

Objective: Minimize total cost

$$\text{Min} \sum_{i \in \mathcal{P}} x_p,$$

Constraints

1. Demand of each item must be fulfilled

$$\sum_{p \in \mathcal{P}} a_{ip} x_p >= d_i, \quad \forall i \in \mathcal{I},$$

2. Non-negativity and integrality constraints

$$x_p \in \mathbb{Z}_+, \quad \forall p \in \mathcal{P}.$$

Check:

$$\sum_{i \in \mathcal{I}} a_{ip} w_i <= W_S. \quad \forall p \in \mathcal{P},$$

# The Knapsack (sub) Problem

Problem description:

- Pick a new 'pattern' from $\mathcal{P}$ with most negative 'reduced cost'
- 'Value' of item $v_i, i \in \mathcal{I}$ (multiplier of demand constraint)

Decision variables:

- $u_i \in \mathcal{I}$: number of times an item $i$ is cut in the (new) pattern

Objective: Minimize reduced cost

$$\text{Min} \quad 1 - \sum_{i \in \mathcal{I}} u_i v_i$$

Constraints

1. The generated pattern must be valid

$$\sum_{i \in \mathcal{I}} u_i w_i <= W_S,$$

2. Non-negativity and integrality constraints

$$u_i \in \mathbb{Z}_+, \quad \forall i \in \mathcal{I}.$$

# Column Generation

**repeat**

Start with a set of 'initial' patterns ($m$) and solve the master problem.

Find the multipliers corresponding to the demand constraints: get $v_i$.

Solve the subproblem (knapsack) and obtain a new cutting pattern.

**until** *the subproblem has a negative objective value*;

# AMPL Modeling Tip 1: Master and subproblem

## Master problem

```
var Cut {PATTERNS} integer >= 0;   #  stocks cut using a pattern

minimize Number:                    # minimize total stock rolls
   sum {p in PATTERNS} Cut[p];

subject to Fill {i in ITEMS}:
   sum {p in PATTERNS} a[i,p] * Cut[p] >= demand[i];
```

## Subproblem

```
var Use {ITEMS} integer >= 0;

minimize Reduced_Cost:
   1 - sum {i in ITEMS} price[i] * Use[i];

subj to Width_Limit:
   sum {i in ITEMS} i * Use[i] <= W_S;
```

# AMPL Modeling Tip 2: Run File

```
model csp.mod; data csp.dat;
problem Cutting_Opt: Cut, Number, Fill;
problem Pattern_Gen: Use, Reduced_Cost, Width_Limit;

repeat {
   solve Cutting_Opt;
   let {i in WIDTHS} price[i] := Fill[i].dual;

   solve Pattern_Gen;
   if Reduced_Cost < -0.00001 then {
      let nPAT := nPAT + 1;
      let {i in WIDTHS} nbr[i,nPAT] := Use[i];
      }
   else break;
   };
```

See `csp.mod`, `csp.dat` and `colgen.ampl`