Exercise 3: Solving the Trust-Region Subproblem

$$\underset{w\in\{0,1\}^p}{\text{minimize}}\ \mathcal{F}^{(k)} + g^{(k)^T}\left(w - w^{(k)}\right) \quad \text{subject to } \|w - w^{(k)}\|_1 \leq \Delta_k,$$

Exercise: Solve TR-subproblem in $\mathcal{O}(N)$ steps; N dimension of w!

1. Re-write trust-region constraint as knapsack-like constraint:

$$\|w - w^{(k)}\|_1 \leq \Delta_k \quad \Leftrightarrow \quad \sum_{w_i^{(k)}=0} w_i - \sum_{w_i^{(k)}=1} w_i \leq \Delta_k - \sum_{w_i^{(k)}=1} 1$$

2. Distinguish four case
   1. $w_i^{(k)} = 0$ and $g_i^{(k)} \geq 0 \Rightarrow$ fix $w_i = 0$ — no reduction
   2. $w_i^{(k)} = 1$ and $g_i^{(k)} \leq 0 \Rightarrow$ fix $w_i = 1$ — no reduction
   3. $w_i^{(k)} = 0$ and $g_i^{(k)} < 0$
   4. $w_i^{(k)} = 1$ and $g_i^{(k)} > 0$

   ... get knapsack problem: $\min_{\hat{w}} \hat{g}^T \hat{w}$ s.t. $\mathbb{1}^T \hat{w} \leq \delta$