Optimization Problems with Equilibrium Constraints

GIAN Short Course on Optimization: Applications, Algorithms, and Computation

Sven Leyffer

Argonne National Laboratory

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Outline

1. Introduction: Stackelberg Games
2. Difficulties with MPECs
3. Stationarity Conditions for MPECs
   - Bouligand and Strong Stationarity
   - Alphabet Soup of Spurious Stationarity

Introduction: Nash Games

Nash Game: non-cooperative equilibrium of several producers

$$z_i^* \in \begin{cases} \underset{z_i}{\operatorname{argmin}} & b_i(\hat{z}) \\ \text{subject to } & c_i(z_i) \geq 0 \\ & z_i \geq 0 \end{cases} \quad \text{producer } i$$

Producer i optimizes own z_i, given other producers choices

- All producers $\hat{z} = (z_1^*, \ldots, z_{i-1}^*, z_i, z_{i+1}^*, \ldots, z_I^*)$
- No shared constraints (otherwise called Nash-Gournot)
- All producers/players are equal

Definition (Nash Equilibrium)

No producer i can improve objective, if other producer's variables, $z_i, \forall j \neq i$, remain unchanged.

Solution of Nash Games

Form first-order optimality conditions for each player ...

$$\text{(NCP)} \qquad \begin{cases} 0 \le \mu \perp \nabla b(z) - \nabla c(z)\lambda \ge 0 \\ 0 \le \lambda \perp c(z) \ge 0 \end{cases}$$

where

- $b(z) = (b_1(z), \ldots, b_k(z))$ & $c(z) = (c_1(z), \ldots, c_k(z))$
- $\perp$ means $\lambda^T c(z) = 0$, either $\lambda_i > 0$ or $c_i(z) > 0$
- Called a nonlinear complementarity problem (NCP)
- Robust large scale solvers exist: e.g. `PATH`

Setting $y = (z, \lambda, \mu)^T$ and $F(y) = (b(z) - \nabla c(z)\lambda,\ c(z))^T$, we can rewrite (NCP) equivalently as

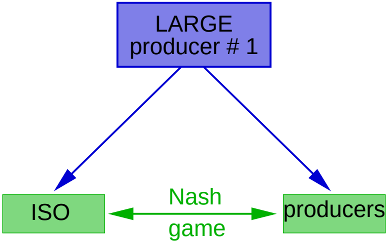
$$0 \le y \ \perp\ F(y) \ge 0$$

... change of notation: y both variables and multipliers!

Stackelberg Games & Bilevel Optimization

Single dominant producer & Nash followers

$$\begin{cases} \underset{x \geq 0, y}{\text{minimize}} & f(x, y) \\ \text{subject to} & c(x, y) = 0 \\ & 0 \leq y \ \perp \ F(x, y) \geq 0 \end{cases}$$

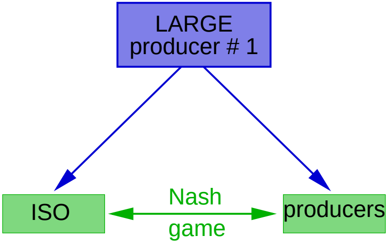


- Nash game $(0 \leq y \ \perp \ F(x, y) = 0)$
  ... parameterized in leader's variables x
- Mathematical Program with Equilibrium Constraints (MPEC)

Bilevel Optimization as MPECs

Single dominant producer & Nash followers equivalent to

$$\begin{cases} \underset{x\geq 0,y}{\text{minimize}} \quad f(x,y) \\ \text{subject to } c(x,y)=0 \\ \qquad\qquad\quad \begin{cases} \min\limits_{y} \; b(y) \\ \text{s.t. } \; d(y,x)\geq 0 \end{cases} \end{cases}$$



- Lower-level problem $(\min \; b(y)$ s.t. $d(y,x)\geq 0)$
  ... parameterized in leader's variables x
- Mathematical Program with Equilibrium Constraints (MPEC)

Example: Optimal Taxation Model

Government sets tax rates, t_g, for certain goods to maximize revenue

- Consumers buy goods to maximize own utility function
- Consumers react to tax rates by changing purchase behavior
- Government is leader ... knows how consumers will react

Assume we have seven goods:

$$\mathcal{G} = \{\text{Beer, Pizza, Movie, Wine, Cheese, Ballet, Leisure}\}$$

... and two classes of consumers

$$\mathcal{C} = \{\text{Students, Professors}\}$$

Example: Optimal Taxation Model

Consumer c buys quantities $q_{c,g} \geq 0$ of goods, $g \in \mathcal{G}$ to

$$\begin{cases} \underset{q}{\text{maximize}} & U_c(q) = \prod_{g \in \mathcal{G}} q_{c,g}^{\alpha_{c,g}} & \text{utility function} \\ \text{subject to} & \sum_{g \in \mathcal{G}} p_g(1 + t_g) q_{c,g} \leq b_c & \text{budget constraint} \end{cases}$$

where $\sum \alpha_{c,g} = 1$, with prices, p_g, and tax-rates, t_g of good $g \in \mathcal{G}$

KKT conditions of consumer c are:

$$-\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in \mathcal{G}: g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c \perp \pi_c \geq 0 \quad \text{and} \quad 0 \leq q_{c,g} \perp \xi_{c,g} \geq 0$$

Example: Optimal Taxation Model

Government maximizes tax revenue subject to consumer actions

$$\max_{t} \sum_{c \in \mathcal{C}} \sum_{g \in \mathcal{G}} t_g q_{c,g} N_c$$

$$\text{s.t.} \quad -\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in \mathcal{G}: g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c \ \perp \ \pi_c \geq 0$$

$$0 \leq q_{c,g} \ \perp \ \xi_{c,g} \geq 0, \qquad \forall c \in \mathcal{C}, \ \forall g \in \mathcal{G}$$

where N_c is the number of consumers in class $c \in \mathcal{C}$

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$$0 \leq q_{c,g} \ \perp \ \xi_{c,g} \geq 0, \qquad \forall c \in \mathcal{C}, \ \forall g \in \mathcal{G}$$

where N_c is the number of consumers in class $c \in \mathcal{C}$

So who gets taxed the most???

The Problem for the Rest of the Day

Mathematical Program with Equilibrium Constraints (MPEC)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \perp F(x,y) \geq 0 \end{cases}$$

- $f : R^p \times R^q \to R$, and $c : R^p \times R^q \to R^m$ smooth
- Complementarity constraint: $F : R^p \times R^q \to R^q$ smooth
  $y_i = 0$ or $F_i(x,y) = 0$ ... $y^T F(x,y) = 0$
- more general $l \leq c(x,y) \leq u$: no problem

MPEC: Economic Applications

- Stackelberg games [Stackelberg, 1952]
- modeling of competition in deregulated electricity markets [Pieper, 2001, Hobbs et al., 2000]
- volatility estimation in American option pricing [Huang and Pang, 1999]
- transportation network design:
  1. toll road pricing: how to set toll levels — leader
  2. consumers move optimally (Wardrop's principle) — followers

  [Hearn and Ramana, 1997, Ferris et al., 1999]

MPEC: Engineering Applications

- design of structures involving friction [Ferris and Tin-Loi, 1999a]
- brittle fracture identification [Tin-Loi and Que, 2002]
- problems in elastoplasticity [Ferris and Tin-Loi, 1999b]
- process engineering models [Rico-Ramirez and Westerberg, 1999, Raghunathan and Biegler, 2002]
- floor planning (design of semi-conductors) [Anjos and Vanelli, 2002]
- obstacle problems (PDE); packaging problems [Outrata et al., 1998]

Outline

Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \ \perp \ F(x,y) \geq 0 \end{cases}$$

Equivalent smooth nonlinear program (NLP):

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = 0 \end{cases}$$

Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

$$
\begin{cases}
\underset{x,y}{\text{minimize}} \;\; f(x,y) \\
\text{subject to } c(x,y) \geq 0 \\
\qquad\qquad\quad 0 \leq y \;\perp\; F(x,y) \geq 0
\end{cases}
$$

Equivalent smooth nonlinear program (NLP):

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\begin{cases}
\underset{x,y}{\text{minimize}} \;\; f(x,y) \\
\text{subject to } c(x,y) \geq 0 \\
\qquad\qquad\quad F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\
\qquad\qquad\quad y^T F(x,y) = 0
\end{cases}
$$

NLP solvers converge slowly, and sometimes fail completely!

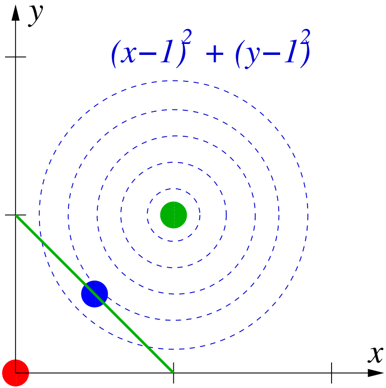
Example of Linear Convergence of SQP

Consider

$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \perp y \ge 0$$

SQP method:

- Start at $(1,1)$



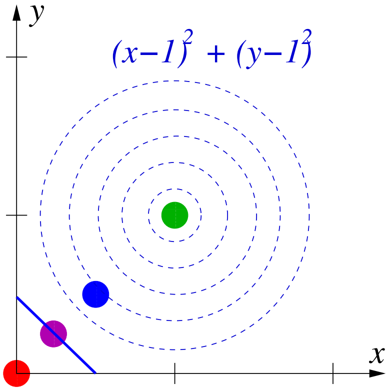
Example of Linear Convergence of SQP

Consider

$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \ \perp \ y \ge 0$$

SQP method:

- Start at $(1, 1)$
- $(x_2, y_2) = (1/2,\ 1/2)$



Example of Linear Convergence of SQP

Consider

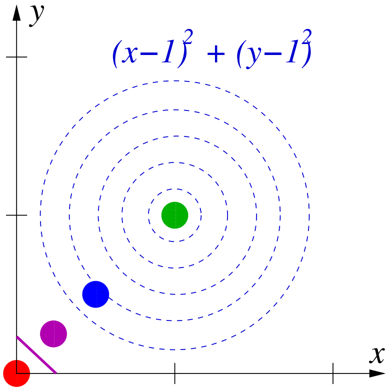
$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \perp y \ge 0$$

SQP method:

- Start at $(1, 1)$
- $(x_2, y_2) = (1/2,\ 1/2)$
- $(x_3, y_3) = (1/2^k,\ 1/2^k)$

... linear convergence to $(0, 0)$

... multipliers $\to \infty$



... not even stationary! $s = (0, 1)$ $s = (1, 0)$ descend!

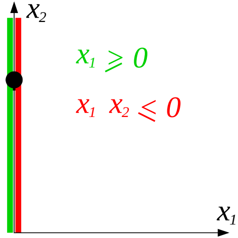
Mangasarian Fromowitz CQ fails

Mangasarian Fromowitz Constraint Qualification at feasible $\hat{x}$:

$$\hat{x}_1 = 0,\ \hat{x}_2 > 0$$

$$\Rightarrow x_1 \geq 0, \quad \text{and} \quad x_2 x_1 \leq 0 \quad \text{active}$$

$$\Rightarrow \text{ MFCQ: } s_1 > 0, \quad \text{and } \hat{x}_2 s_1 < 0$$



MFCQ is important (minimalist) stability assumption for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set unbounded ... $\nabla^2 \mathcal{L}$ may blow up
2. Constraint gradients linearly dependent ... ill-conditioned steps
3. Central path does not exist ... IPMs may not work at all!

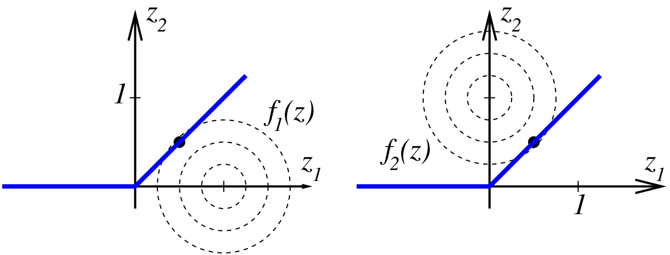
Dependent Constraints and Unbounded Multiplier Sets

Consider the two QPECs

$$\begin{cases} \underset{z}{\text{minimize}} & f_i(x,y) \\ \text{subject to } 0 \le y \ \perp \ y - x \ge 0 \end{cases}$$

with $f_1(z) = (x-1)^2 + y^2$ and $f_2(z) = x^2 + (y-1)^2$

Solution at $(x,y)^* = (1/2, 1/2)^T$



Dependent Constraints and Unbounded Multiplier Sets

Equivalent NLP of QPECs is

$$\left\{\begin{array}{lll} \underset{z}{\text{minimize}} & f_i(z) & \text{multiplier} \\ \text{subject to} & y \geq 0 & \nu \geq 0 \\ & y - x \geq 0 & \lambda \geq 0 \\ & y\,(y - x) \leq 0 & \xi \geq 0. \end{array}\right.$$

with KKT conditions:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda^* \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \xi^* \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

... active constraint normals are clearly dependent!

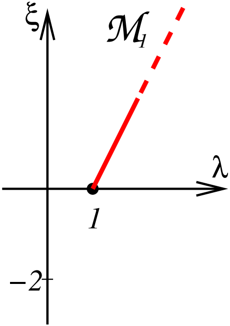
Dependent Constraints and Unbounded Multiplier Sets

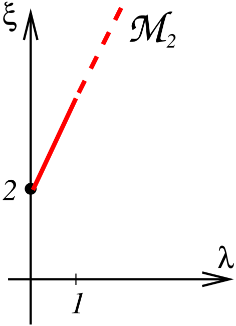
Since $y^* = \frac{1}{2} > 0$ we see $\nu^* = 0$, and multiplier sets ...

$$\mathcal{M}_1 \;=\; \left\{(\lambda,\xi) \mid \xi \geq 0,\ \lambda + \tfrac{1}{2}\xi = 1\right\}$$

$$\mathcal{M}_2 \;=\; \left\{(\lambda,\xi) \mid \lambda \geq 0,\ -\lambda + \tfrac{1}{2}\xi = 1\right\},$$

... are unbounded





Inconsistent Linearizations

MPECs can have inconsistent linearizations arbitrarily close to stationary point

$$\begin{cases} \underset{z}{\text{minimize}} & x + y \\ \text{subject to} & y^2 \geq 1 \\ & 0 \leq x \perp y \geq 0. \end{cases}$$

Nice solution: $(x, y)^* = (0, 1)^T$ multipliers $\lambda^* = 0.5$
Linearize at $(\hat{x}, \hat{y}) = (\epsilon, 1 - \delta)^T$ with $\epsilon, \delta > 0$:

$$(1-\delta)^2 + 2(1-\delta)(y - (1-\delta)) \geq 1 \quad \Rightarrow \quad y \geq \frac{1 + (1-\delta)^2}{2(1-\delta)} > 1$$

and

$$(1-\delta)\epsilon + (1-\delta)(x - \epsilon) + \epsilon(y - (1-\delta)) \leq 0 \quad \Rightarrow \quad y \leq 1 - \delta < 1$$

How Else Can We Solve MPECs?

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = 0 \end{cases}$$

Goal

Want to use the good NLP solvers, such as IPM, SQP, SLQP, ...
Trouble caused by too many dependent active constraints:
$F(x,y) = 0$ and $y = 0$ and $y^T F(x,y) = 0$ ... remove one!

Two alternative approaches that use NLP solvers:

1. Relax the complementarity constraint
2. Penalize the complementarity constraint

NLP-Based Relaxation Approach to MPECs

Formulate a relaxed NLP

$$(\text{R-NLP}(\rho)) \qquad \begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = \rho \end{cases}$$

... for $\rho \searrow 0$

Given initial $\rho > 0$

repeat

  Solve (R-NLP(ρ)) for (x^ρ, y^ρ)

  Reduce $\rho := \rho/4$

until (x^ρ, y^ρ) *is solution of MPEC*;

NLP-Based Penalization Approach to MPECs

Formulate a penalized NLP

$$
\text{(P-NLP}(\rho)) \qquad \begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) + \pi \| y^T F(x,y) \| \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \end{cases}
$$

... for $\pi \nearrow 0$ ... problem satisfies MFCQ!

Given initial $\pi > 0$

repeat

  Solve (P-NLP(π)) for (x^π, y^π)

  Reduce $\pi := 4\pi$

until (x^π, y^π) *is solution of MPEC*;

Relaxation and penalization are loosely related ...

An Even Simpler Trick Seems to Work

Consider an alternative (lazy) reformulation of MPEC

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \perp F(x,y) \geq 0 \end{cases}$$

Introduce slack variables s:

- Write $F(x,y) = s$ as nonlinear equation
- Simplify the complementarity to bilinear inequality $y^T s \leq 0$
- Equivalent, because $s, y \geq 0$ ... solvers satisfy bounds easily

Equivalent smooth nonlinear program (NLP):

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) = s, \quad s \geq 0, \quad y \geq 0 \quad \text{and} \quad y^T s \leq 0 \end{cases}$$

... more in the next lecture!

Outline

MPEC Bouligand-Stationarity

Definition (MPEC B-Stationarity)

(x^*, y^*) is *B-stationary*, iff $d = 0$ solves LPEC

$$\begin{aligned} \underset{d}{\text{minimize}} \quad & {g^*}^T d \\ \text{subject to } & c^* + {A^*}^T d \geq 0, \\ & 0 \leq y^* + d_y \perp F^* + {B^*}^T d \geq 0, \end{aligned}$$

where $g^* = \nabla f(x^*, y^*), \quad A^* = \nabla c(x^*, y^*), \quad B^* = \nabla F(x^*, y^*)$

B-stationarity is a structural stationarity condition

- Applies stationarity to nonlinear functions
- Retains structure of the problem $\Rightarrow$ strong result
- Absence of feasible descend directions!

  ... similar to LP being stationary for NLP

MPEC Strong-Stationarity

- (x^*, y^*) is weakly-stationary, iff $\exists$ λ, μ, and ν:

$$\begin{aligned} g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} &= 0, \\ 0 \le c^* \ \perp \ \lambda &\ge 0, \\ 0 \le y^* \ \perp \ F^* &\ge 0. \end{aligned}$$

where $\nu \ \perp \ y^*$ and $\mu \ \perp \ F(x,y)$ ... μ, ν unrestricted

- Degenerate complementarity conditions:

$$\mathcal{D}(z) := \left\{ i : y_i = 0 = F_i(z) \right\}$$

- (x^*, y^*) is strongly-stationary iff

$$\mu_i \ge 0, \ \nu_i \ge 0, \ \forall i \in \mathcal{D}^*$$

... equivalent to KKT conditions of equivalent NLP

Alphabet Soup of Spurious Stationarity

(x^*, y^*) is weakly-stationary, iff $\exists\ \lambda$, μ, and ν:

$$\begin{aligned} g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} &= 0, \\ 0 \le c^* \ \perp \ \lambda &\ge 0, \\ 0 \le y^* \ \perp \ F^* &\ge 0. \end{aligned}$$

where $\nu \ \perp \ y^*$ and $\mu \ \perp \ F(x, y)$

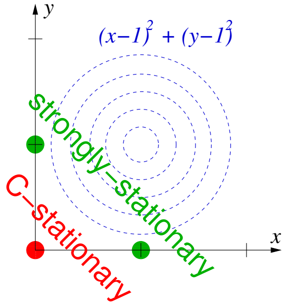
Degenerate complementarity: $\mathcal{D}(z) := \{ i : y_i = 0 = F_i(z) \}$

- A-stationary, iff $\mu_i \ge 0$ or $\nu_i \ge 0, \ \forall i \in \mathcal{D}^*$
- C-stationary, iff $\mu_i \nu_i \ge 0 \ \forall i \in \mathcal{D}^*$
- M-stationary, iff $\left(\mu_i > 0 \text{ and } \nu_i > 0 \right)$ or $\mu_i \nu_i = 0, \ \forall i \in \mathcal{D}^*$

all have trivial descend directions

Spuriousness of C-Stationarity

Consider min $(x-1)^2 + (y-1)^2$ subject to $0 \leq y \perp x \geq 0$:

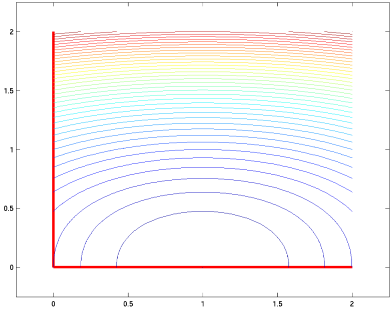


(0, 0) C-stationary: $\mu = \nu = -2 < 0$!!!

$\Rightarrow \exists$ descend directions

Spuriousness of A/M-Stationarity

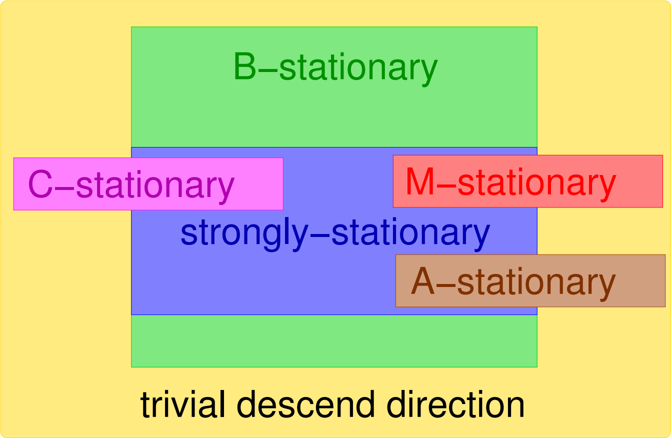
Consider min $(x-1)^2 + y^3 + y^2$ subject to $0 \leq y \ \perp \ x \geq 0$



$(0,0)$ M/A-stationarity: $\mu = 0, \nu = -2 < 0$!!!

$\Rightarrow$ *exists* descend directions

Alphabet Soup of Stationarity



A/B/C/M/S-stationarity equivalent, iff $\mathcal{D}^* = \emptyset$

What Have We Learned?

Complementarity constraints are important class of problems

- Arise in many applications ... useful modeling paradigm
- Students should pay more taxes than their professors

MPECs are a challenging class of problems

- Violate MFCQ $\Rightarrow$ unbounded multipliers, infeasible linearizations
- NLP solvers can fail

Extended optimality conditions

- B-stationarity is the best ... and most difficult
- Strong stationarity is good ... but does not always hold
- Many useless stationarity concepts: A-, C-, L-, M-, W- ...

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