# Mixed-Integer Nonlinear Optimization: Convex MINLPs

## GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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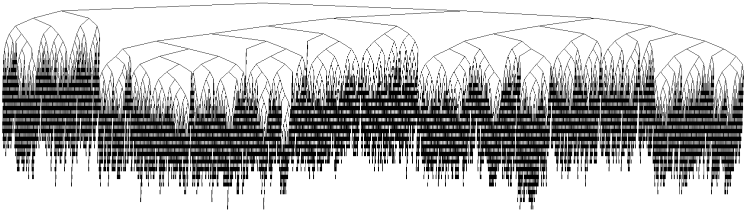
# Outline

1. Multi-Tree Methods
2. Single-Tree Methods
3. Presolve for MINLP

# Montivation MINLP Trees are Huge



Synthesis MINLP B&B Tree: 10000+ nodes after 360s

- Requires solution of thousands of NLPs
  QP solves can be good alternative
- Can we have even faster solves at nodes?
  Consider MILP solvers to search tree ...

# Multi-Tree Methods

MILP solvers much better developed than MINLP

- LPs are easy to hot-start
- Decades of investment into software
- MILPs much easier; e.g. no need for constraint qualifications

⇒ developed methods that exploit this technology

Multi-Tree Methods

- Outer approximation [Duran and Grossmann, 1986]
- Benders decomposition [Geoffrion, 1972]
- Extended cutting plane method [Westerlund and Pettersson, 1995]

... solve a sequence of MILP (and NLP) problems

Multi-tree methods evaluate functions "only" at integer points!

# Multi-Tree Methods

Recall the $\eta$-MINLP formulation

$$\begin{cases} \underset{\eta,x}{\text{minimize}} & \eta, \\ \text{subject to} & f(x) \leq \eta, \\ & c(x) \leq 0, \\ & x \in \mathcal{X}, \\ & x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}. \end{cases}$$

where we have "linearized" the objective: $\eta \geq f(x)$

Use $\eta$-MINLP in this section

# Outer Approximation

Mixed-Integer Nonlinear Program (MINLP)

$$\underset{x}{\text{minimize}}\ f(x) \quad \text{subject to } c(x) \leq 0,\ x \in \mathcal{X},\ x_i \in \mathbb{Z}\ \forall\ i \in \mathcal{I}$$

NLP subproblem for fixed integers $x_{\mathcal{I}}^{(j)}$:

$$\text{NLP}(x_{\mathcal{I}}^{(j)}) \begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to } & c(x) \leq 0 \\ & x \in \mathcal{X} \quad \text{and } x_{\mathcal{I}} = x_{\mathcal{I}}^{(j)}, \end{cases}$$

with solution $x^{(j)}$.

If $(\text{NLP}(x_{\mathcal{I}}^{(j)}))$ infeasible then solve feasibility problem ...

# Outer Approximation

Convexity of $f$ and $c$ implies that

**Lemma (Supporting Hyperplane)**

*Linearization about solution $x^{(j)}$ of $(NLP(x_{\mathcal{I}}^{(j)}))$*

$$(OA) \quad \eta \geq f^{(j)} + \nabla f^{(j)^T}(x - x^{(j)}) \quad \textit{and} \quad 0 \geq c^{(j)} + \nabla c^{(j)^T}(x - x^{(j)}),$$

*are outer approximations of the feasible set of $\eta$-MINLP.*

**Lemma (Feasibility Cuts)**

*If $(NLP(x_{\mathcal{I}}^{(j)}))$ infeasible, then (OA) cuts off $x_{\mathcal{I}} = x_{\mathcal{I}}^{(j)}$.*

# Outer Approximation

Mixed-Integer Nonlinear Program ($\eta$-MINLP)

$$\min_x \ \eta \quad \text{s.t. } \eta \geq f(x),\ c(x) \leq 0,\ x \in \mathcal{X},\ x_i \in \mathbb{Z}\ \forall\ i \in \mathcal{I}$$

Define index set of all possible feasible integers, $\mathcal{F}$

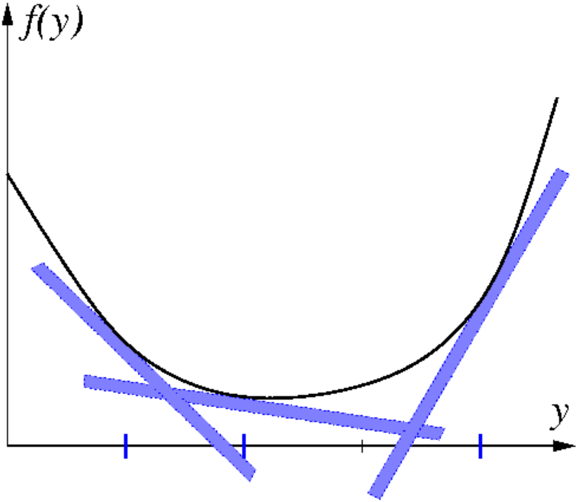
$$\mathcal{F} := \left\{ x^{(j)} \in \mathcal{X} : x^{(j)} \text{ solves } (\text{NLP}(x_{\mathcal{I}}^{(j)})) \quad \text{or } (\text{F}(x_{\mathcal{I}}^{(j)})) \right\}.$$

... boundedness of $\mathcal{X}$ implies $|\mathcal{F}| < \infty$

Construct equivalent OA-MILP (outer approximation MILP)

$$\begin{cases} \underset{\eta,x}{\text{minimize}} & \eta, \\ \text{subject to} & \eta \geq f^{(j)} + \nabla f^{(j)^T}(x - x^{(j)}),\ \forall x^{(j)} \in \mathcal{F} \\ & 0 \geq c^{(j)} + \nabla c^{(j)^T}(x - x^{(j)}),\ \forall x^{(j)} \in \mathcal{F} \\ & x \in \mathcal{X}, \\ & x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}. \end{cases}$$

# Outer Approximation in Less Than 1000 Words



# Outer Approximation Algorithm

Solving OA-MILP clearly not sensible; define upper bound as

$$U^k := \min_{j \leq k} \left\{ f^{(j)} \mid (\text{NLP}(x_{\mathcal{I}}^{(j)})) \text{ is feasible } \right\}.$$

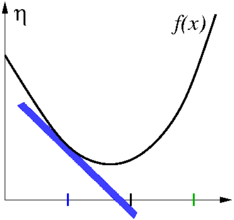
Define relaxation of OA-MILP, using $\mathcal{F}^k \subset \mathcal{F}$, with $\mathcal{F}^0 = \{0\}$

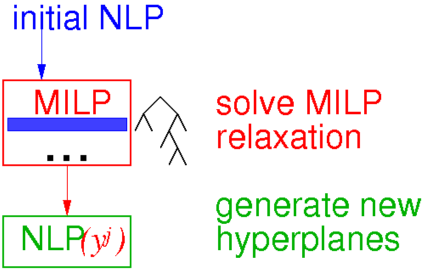
$$M(\mathcal{F}^k) \begin{cases} \underset{\eta, x}{\text{minimize}} & \eta, \\ \text{subject to} & \eta \leq U^k - \epsilon \\ & \eta \geq f^{(j)} + \nabla f^{(j)^T}(x - x^{(j)}), \ \forall x^{(j)} \in \mathcal{F}^k \\ & 0 \geq c^{(j)} + \nabla c^{(j)^T}(x - x^{(j)}), \ \forall x^{(j)} \in \mathcal{F}^k \\ & x \in \mathcal{X}, \\ & x_i \in \mathbb{Z}, \ \forall i \in \mathcal{I}. \end{cases}$$

... build up better OA $\mathcal{F}^k$ iteratively for $k = 0, 1, \ldots$

# Outer Approximation Algorithm

Alternate between solve NLP($y_j$) and MILP relaxation




MILP $\Rightarrow$ lower bound; NLP $\Rightarrow$ upper bound

... convergence follows from convexity & finiteness

# Outer Approximation Algorithm

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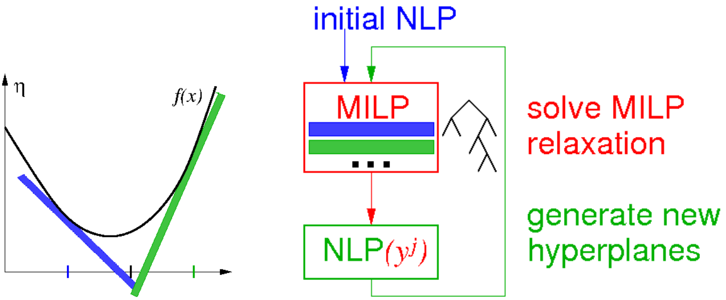



MILP ⇒ lower bound; NLP ⇒ upper bound

... convergence follows from convexity & finiteness

# Outer Approximation Algorithm

**Outer approximation** ;

Given $x^{(0)}$, choose tol $\epsilon > 0$, set $U^{-1} = \infty$, set $k = 0$, and $\mathcal{F}^{-1} = \emptyset$. ;

**repeat**

Solve (NLP($x_{\mathcal{I}}^{(j)}$)) or (F($x_{\mathcal{I}}^{(j)}$)); solution $x^{(j)}$.;

**if** *(NLP($x_{\mathcal{I}}^{(j)}$)) feasible & $f^{(j)} < U^{k-1}$* **then**

Update best point: $x^* = x^{(j)}$ and $U^k = f^{(j)}$.;

**else**

Set $U^k = U^{k-1}$.;

**end**

Linearize $f$ and $c$ about $x^{(j)}$ and set $\mathcal{F}^k = \mathcal{F}^{k-1} \cup \{j\}$. ;

Solve $(M(\mathcal{F}^k))$, let solution be $x^{(k+1)}$ & set $k = k + 1$. ;

**until** *MILP $(M(\mathcal{F}^k))$ is infeasible*;

# Outer Approximation Algorithm

## Theorem (Convergence of Outer Approximation)

*Let Assumptions A1-A3 hold, then outer approximation terminates finitely at optimal solution of MINLP or indicates it is infeasible.*

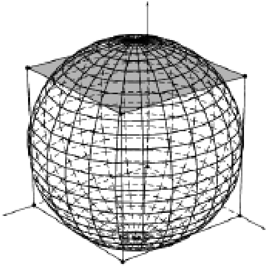
**Outline of Proof.**

- Optimality of $x^{(j)}$ in (NLP($x_{\mathcal{I}}^{(j)}$))
  $\Rightarrow \eta \geq f^{(j)}$ for feasible point of $(M(\mathcal{F}^k))$
  ... ensures finiteness, since $\mathcal{X}$ compact
- Convexity $\Rightarrow$ linearizations are supporting hyperplanes
  ... ensures optimality

# Worst Case Example of Outer Approximation

[Hijazi et al., 2010] construct infeasible MINLP:

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & \sum_{i=1}^{n} \left( y_i - \frac{1}{2} \right)^2 \leq \frac{n-1}{4} \\ & y \in \{0,1\}^n \end{aligned}$$



Intersection of ball of radius $\frac{\sqrt{n-1}}{2}$ with unit hypercube.

**Lemma**

*OA cannot cut more than one vertex of the hypercube*
*MILP master problem feasible for any $k < 2^n$ OA cuts*

**Theorem**

*OA visits all $2^n$ vertices*

# Benders Decomposition

Can derive Benders cut from outer approximation:

- Take optimal multipliers $\lambda^{(j)}$ of (NLP($x_{\mathcal{I}}^{(j)}$))
- Sum outer approximations

$$
\begin{array}{rrl}
 & \eta \geq & f^{(j)} + \nabla f^{(j)^T}(x - x^{(j)}) \\
+ & \lambda^{(j)^T} (\; 0 \geq & c^{(j)} + \nabla c^{(j)^T}(x - x^{(j)}) \;) \\
\hline
 & \eta \geq & f^{(j)} + \nabla_{\mathcal{I}} \mathcal{L}^{(j)^T}(x_{\mathcal{I}} - x_{\mathcal{I}}^{(j)})
\end{array}
$$

- Using KKT conditions wrt continuous variables $x_C$:
  $0 = \nabla_C \mathcal{L}^{(j)} = \nabla_C f + \nabla_C c \lambda^{(j)}$ & $\lambda^{(j)^T} c^{(j)} = 0$
  ... eliminates continuous variables, $x_C$

Benders cut only involves integer variables $x_{\mathcal{I}}$.
Can write cut as $\eta \geq f^{(j)} + \mu^{(j)^T}(x_{\mathcal{I}} - x_{\mathcal{I}}^{(j)})$,
where $\mu^{(j)}$ multiplier of $x = x_{\mathcal{I}}^{(j)}$ in (NLP($x_{\mathcal{I}}^{(j)}$))

# Benders Decomposition

For MINLPs with convex problems functions $f$, $c$, we can show:

1. Benders cuts are weaker than outer approximation
   - Benders cuts are linear combination of OA
2. Outer Approximation & Benders converge finitely
   - Functions $f$, $c$ convex $\Rightarrow$ OA cuts are outer approximations
   - OA cut derived at optimal solution to NLP subproblem
     $\Rightarrow$ $\nexists$ feasible descend directions
     ... every OA cut corresponds to first-order condition
   - Cannot visit same integer $x_{\mathcal{I}}^{(j)}$ more than once

   $\Rightarrow$ terminate finitely at optimal solution

Readily extended to situations where (NLP($x_{\mathcal{I}}^{(j)}$)) not feasible.

# Summary of Multi-Tree Methods

Three Classes of Multi-Tree Methods (did not discuss ECP)

1. Outer approximation based on first-order expansion
2. Benders decomposition linear combination of OA cuts
3. Extended cutting plane method: avoids NLP solves

Common Properties of Multi-Tree Methods

- Only need to solve final MILP to optimality
  ... can terminate MILP early ... adding more NLPs
- Can add cuts from incomplete NLP solves
- Worst-case example for OA also applies for Benders and ECP
- No warm-starts for MILP ... expensive tree-search

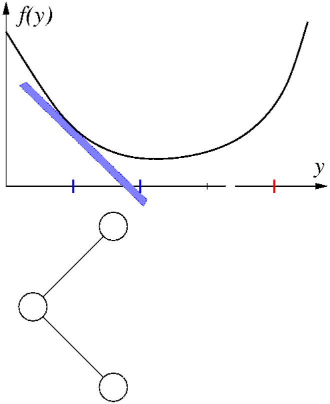
... motivates single-tree methods next ...

# Outline

# LP/NLP-Based Branch-and-Bound

Aim: avoid solving expensive MILPs

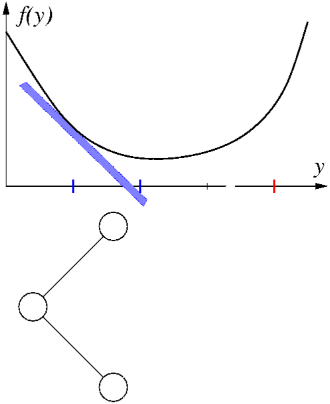
- Form MILP outer approximation



# LP/NLP-Based Branch-and-Bound

Aim: avoid solving expensive MILPs

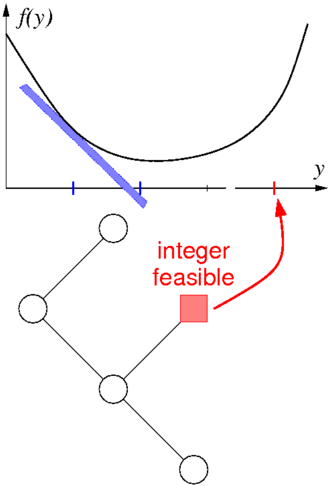
- Form MILP outer approximation
- Take initial MILP tree



# LP/NLP-Based Branch-and-Bound

Aim: avoid solving expensive MILPs

- Form MILP outer approximation
- Take initial MILP tree
- interrupt MILP, when new
  integral $x_I^{(j)}$ found
  $\Rightarrow$ solve NLP($x_I^{(j)}$) get $x^{(j)}$



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- Form MILP outer approximation
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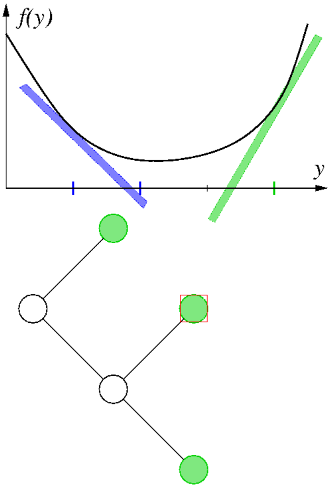
- linearize $f$, $c$ about $x^{(j)}$
  $\Rightarrow$ add linearization to tree

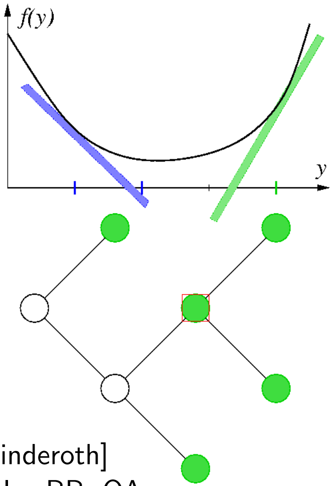


# LP/NLP-Based Branch-and-Bound

Aim: avoid solving expensive MILPs

- Form MILP outer approximation
- Take initial MILP tree
- interrupt MILP, when new integral $x_I^{(j)}$ found
  $\Rightarrow$ solve NLP($x_I^{(j)}$) get $x^{(j)}$
- linearize $f$, $c$ about $x^{(j)}$
  $\Rightarrow$ add linearization to tree
- continue MILP tree-search

... until lower bound $\geq$ upper bound




Software:
FilMINT: FilterSQP + MINTO [L & Linderoth]
BONMIN: IPOPT + CBC [IBM/CMU] also BB, OA

# Branch-and-Cut in MINOTAUR

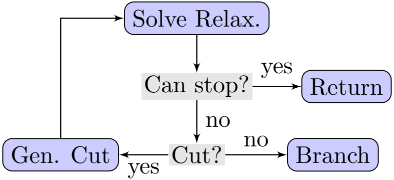
Suppose we need a branch-and-cut solver.

**Node Relaxer**

Obtain linear relaxation in root node.

Only CxLinHandler

**Node Processor**




CxLinHandler
IntVarHandler

**Brancher**

Pick a fractional variable.

Only IntVarHandler

```
relax() {
// Solve NLP
// get Linearization at sol.
}
bool isFeasible() {
// check non-linear constraints
}
```

```
separate() {
// solve NLP
// get Linearization at sol.
}
cand* findBrCandidates() {
// empty
}
```

# LP/NLP-Based Branch-and-Bound

Algorithmic refinements, e.g. [Abhishek et al., 2010]

- Advanced MILP search and cut management techniques
  ... remove "old" OA cuts from LP relaxation ⇒ faster LP
- Generate cuts at non-integer points: ECP cuts are cheap
  ... generate cuts early (near root) of tree
- Strong branching, adaptive node selection & cut management
  - Fewer nodes, if we add more cuts (e.g. ECP cuts)
  - More cuts make LP harder to solve
    ⇒ remove outdated/inactive cuts from LP relaxation

  ... balance OA accuracy with LP solvability
- Compress OA cuts into Benders cuts can be OK

Interpret as hybrid algorithm, [Bonami et al., 2008]

Benders and ECP versions are also possible.

# Outline

# Presolve for MINLP

Presolve plays key role in MILP solvers

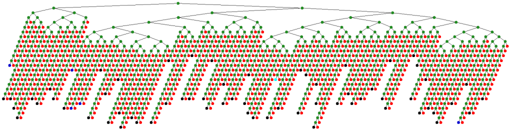
- Bound tightening techniques
- Checking for duplicate rows
- Fixing or removing variables
- Identifying redundant constraints

... creates tighter LP/NLP relaxations $\Rightarrow$ smaller trees!

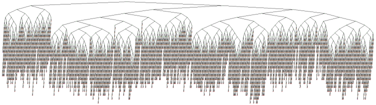
... some presolve in AMPL, but no nonlinear presolve

# What Could Go Wrong in MINLP?

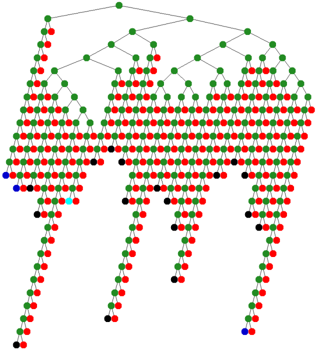
Syn20M04M : a synthesis design problem in chemical engineering
Problem size: 160 Integer Variables, 56 Nonlinear constraints



1000+ nodes after solving for 75s



5000+ nodes after solving for 200s



250+ nodes after solving for 45s

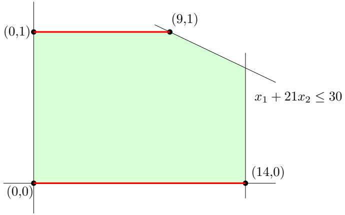
| Solver | CPU | Nodes |
|---|---|---|
| Bonmin | >2h | >149k |
| MINLPBB | >2h | >150k |
| Minotaur | >2h | >264k |

# Improving Coefficients: An Example

$$
\begin{aligned}
(1) \quad x_1 + 21x_2 &\le 30 \\
0 \le x_1 &\le 14 \\
x_2 &\in \{0, 1\}
\end{aligned}
$$

# Improving Coefficients: An Example

$$(1) \quad x_1 + 21x_2 \le 30$$
$$0 \le x_1 \le 14$$
$$x_2 \in \{0, 1\}$$




**If $x_2 = 0$**

$$x_1 + 0 \le 30$$

(1) is loose.

**If $x_2 = 1$**

$$x_1 \le 9$$

(1) is tight.

# Improving Coefficients: An Example

(1) $x_1 + 21x_2 \leq 30$

$0 \leq x_1 \leq 14$

$x_2 \in \{0, 1\}$

**If $x_2 = 0$**

$x_1 + 0 \leq 30$

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**If $x_2 = 1$**

$x_1 \leq 9$

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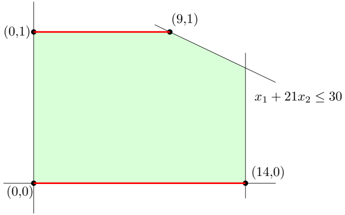
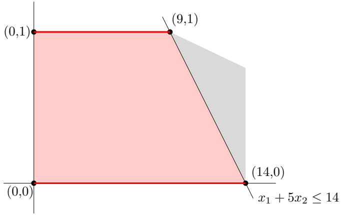
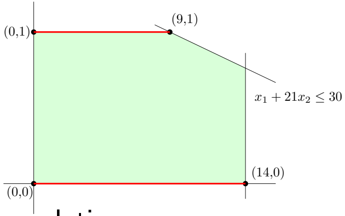


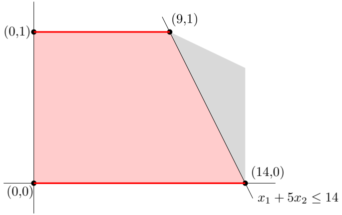



# Improving Coefficients: An Example

$$(1) \quad x_1 + 21x_2 \leq 30$$
$$0 \leq x_1 \leq 14$$
$$x_2 \in \{0,1\}$$

| If $x_2 = 0$ | If $x_2 = 1$ |
|---|---|
| $x_1 + 0 \leq 30$ | $x_1 \leq 9$ |
| (1) is loose. | (1) is tight. |




Reformulation:

$$(2) \quad x_1 + 5x_2 \leq 14$$
$$0 \leq x_1 \leq 14$$
$$x_2 \in \{0,1\}$$

| If $x_2 = 0$ | If $x_2 = 1$ |
|---|---|
| $x_1 + 0 \leq 14$ | I $x_1 \leq 9$ |
| (2) is tight. | (2) is tight. |

(1) and (2) equivalent. But relaxation of (2) is tighter.

# Improving Coefficients: Linear to Nonlinear

$$
\begin{aligned}
c(x_1, x_2, \ldots, x_k) &\le M(1 - x_0) \\
l_i \le x_i &\le u_i, \quad i = 1, \ldots, k \\
x_0 &\in \{0, 1\}
\end{aligned}
$$

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\end{aligned}
$$

- If $c(x_1, x_2, \dots, x_k) \le M(1 - 0)$, is loose, tighten it!

$$
\begin{aligned}
\text{Let } c^u = \max_x \; & c(x_1, \dots, x_k) \qquad \text{(MAX-c)} \\
\text{s.t.} \; & l_i \le x_i \le u_i, \quad i = 1, \dots, k
\end{aligned}
$$

- If $c^u < M$, then tighten: $c(x_1, \dots, x_k) \le c^u(1 - x_0)$

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$$

- If $c^u < M$, then tighten: $c(x_1, \ldots, x_k) \le c^u(1 - x_0)$
- (MAX-c) is a nonconvex NLP ... time-consuming
- Upper bound on (MAX-c) will also tighten
- Trade-off between time and quality of bound: Fast or Tight!

# Improving Coefficients: Using Implications

$$
\begin{aligned}
c(x_1, x_2, \ldots, x_k) &\le M(1 - x_0), \\
l_i \le x_i &\le u_i, \quad i = 1, \ldots, k, \\
x_0 &\in \{0, 1\}.
\end{aligned}
$$

- Often, $x_0$, $x_i$ also occur in other constraints of MINLP. e.g.

$$
\begin{aligned}
c(x_1, x_2, \ldots, x_k) &\le M(1 - x_0) \\
0 \le x_1 &\le M_1 x_0 \\
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&\ldots \\
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# Improving Coefficients: Using Implications

$$c(x_1, x_2, \ldots, x_k) \le M(1 - x_0),$$
$$l_i \le x_i \le u_i, \quad i = 1, \ldots, k,$$
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$$\ldots$$
$$x_0 \in \{0, 1\}$$

- $x_0 = 0 \Rightarrow x_1 = x_2, \ldots = x_k = 0$. (Implications)
- If $c(0, \ldots, 0) < M$, then we can tighten.

# Improving Coefficients: Using Implications

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$$

- $x_0 = 0 \Rightarrow x_1 = x_2, \ldots = x_k = 0$. (Implications)
- If $c(0, \ldots, 0) < M$, then we can tighten.
- No need to solve (MAX-c). Fast and Tight.

# Presolve for MINLP

Advanced functions of presolve (Reformulating):

- Improve coefficients.
- Disaggregate constraints.
- Derive implications and conflicts.

Basic functions of presolve (Housekeeping):

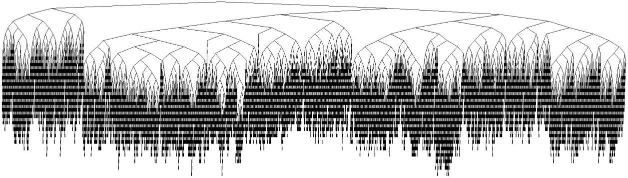
- Tighten bounds on variables and constraints.
- Fix/remove variables.
- Identify and remove redundant constraints.
- Check duplicacy.

Popular in Mixed-Integer Linear Optimization [Savelsbergh, 1994]

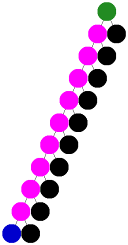
# Presolve for MINLP: Computational Results

`Syn20M04M` from `egon.cheme.cmu.edu`

| | No Presolve | Basic Presolve | Full Presolve |
|---|---|---|---|
| Variables: | 420 | 328 | 292 |
| Binary Vars: | 160 | 144 | 144 |
| Constraints: | 1052 | 718 | 610 |
| Nonlin. Constr: | 56 | 56 | 56 |
| Bonmin(sec): | >7200 | NA | NA |
| Minotaur(sec): | >7200 | >7200 | 2.3 |



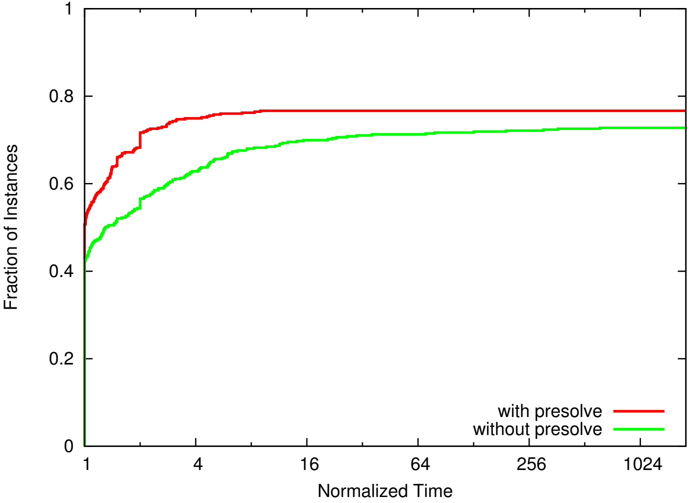
Minotaur, no presolve: 10000+ nodes after solving for 360s



Full Presolve

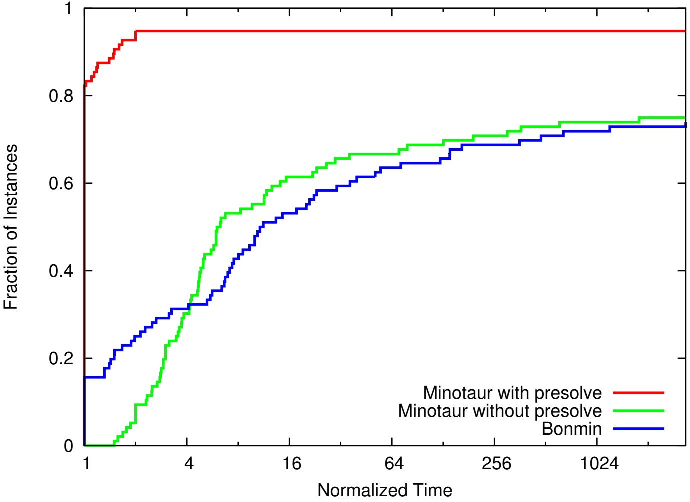
Why does no one else do this?

# Presolve for MINLP: Results




Time taken in Branch-and-Bound on all 463 instances.

# Presolve for MINLP: Results




Time for B&B on 96 RSyn-X and Syn-X instances.

# Presolve for MINLP: Constraint Disaggregation

Nonlinear disaggregation [Tawarmalani and Sahinidis, 2005]

$$S := \{x \in \mathbb{R}^n : c(x) = h(g(x)) \leq 0\},$$

$g : \mathbb{R}^n \to \mathbb{R}^p$ smooth convex;
$h : \mathbb{R}^p \to \mathbb{R}$ smooth, convex, and nondecreasing
$\Rightarrow c(x)$ smooth convex

Disaggregated formulation: introduce $y = g(x) \in \mathbb{R}^p$

$$S_d := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : h(y) \leq 0, \; y \geq g(x)\right\}.$$

**Lemma**

*$S$ is projection of $S_d$ onto $x$.*

# Presolve for MINLP: Constraint Disaggregation

Consider

$$S := \{x \in \mathbb{R}^n : c(x) = h(g(x)) \le 0\},$$

and

$$S_d := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : h(y) \le 0,\ y \ge g(x)\right\}.$$

**Theorem**

*Any outer approximation of $S_d$ is stronger than OA of $S$*

Given $\mathcal{X}^k := \left\{x^{(1)}, \ldots, x^{(k)}\right\}$ construct OA for $S$, $S_d$:

$$S^{oa} := \left\{x : c^{(l)} + \nabla c^{(l)^T}(x - x^{(l)}) \le 0,\ \forall x^{(l)} \in \mathcal{X}^k\right\}$$

$$S_d^{oa} := \left\{(x, y) : h^{(l)} + \nabla h^{(l)^T}(y - g(x^{(l)})) \le 0,\right.$$
$$\left. y \ge g^{(l)} + \nabla g^{(l)^T}(x - x^{(l)}),\ \forall x^{(l)} \in \mathcal{X}^k\right\},$$

[Tawarmalani and Sahinidis, 2005] show $S_d^{oa}$ stronger than $S^{oa}$

# Presolve for MINLP: Constraint Disaggregation

[Hijazi et al., 2010] study

$$\left\{ x : c(x) := \sum_{j=1}^{q} h_j(a_j^T x + b_j) \leq 0 \right\}$$

where $h_j : \mathbb{R} \to \mathbb{R}$ are smooth and convex

Disaggregated formulation: introduce $y \in \mathbb{R}^q$

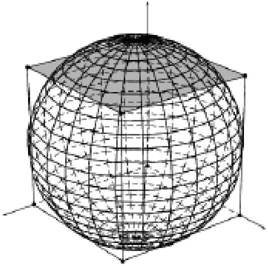
$$\left\{ (x, y) : \sum_{j=1}^{q} y_j \leq 0, \quad \text{and} \quad y_j \geq h_j(a_j^T x + b_j) \right\}$$

can be shown to be tighter

# Recall: Worst Case Example of OA

Apply disaggregation to [Hijazi et al., 2010] example:

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & \sum_{i=1}^{n} \left(x_i - \frac{1}{2}\right)^2 \le \frac{n-1}{4} \\ & x \in \{0,1\}^n \end{aligned}$$



Intersection of ball of radius $\frac{\sqrt{n-1}}{2}$ with unit hypercube.

Disaggregate $\sum \left(x_i - \frac{1}{2}\right)^2 \le \frac{n-1}{4}$ as

$$\sum_{i=1}^{n} y_i \le 0 \quad \text{and} \quad \left(x_i - \frac{1}{2}\right)^2 \le y_i$$

# Presolve for MINLP: Constraint Disaggregation

[Hijazi et al., 2010] disaggregation on worst-case example of OA

- Linearize around $x^{(1)} \in \{0,1\}^n$ and complement $x^{(2)} := e - x^{(1)}$, where $e = (1, \dots, 1)$
- OA of disaggregated constraint is

$$\sum_{i=1}^{n} y_i, \quad \text{and} \quad x_i - \tfrac{3}{4} \leq y_i, \quad \text{and} \quad \tfrac{1}{4} - x_i \leq y_i,$$

- Using $x_i \in \{0,1\}$ implies $z_i \geq 0$, implies $\sum z_i \geq \frac{n}{4} > \frac{n-1}{4}$

$\Rightarrow$ OA-MILP master of $x^{(1)}$ and $x^{(2)}$ is infeasible.

... terminate in two iterations

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