Convergence of Sum-Up Rounding Schemes for the Electromagnetic Cloak Problem*

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Abstract. We consider the design of an electromagnetic cloak from an integer programming point of view. The problem can be modeled as a PDE-constrained optimization problem with integer-valued control inputs that are distributed in the computational domain. A first-discretize-then-optimize approach results in a large-scale mixed-integer nonlinear program that is in general intractable because of the large number of integer variables that arise from the discretization. Instead, we propose an efficient algorithm that approximates the local infima of the underlying non convex infinite-dimensional problem arbitrarily close without the need to solve the discretized finite-dimensional integer programs to optimality. We optimize the continuous relaxations of the approximations and then apply the sum-up rounding methodology to obtain integer-valued controls. These controls are shown to converge and exhibit the desired approximation properties under suitable refinements of the discretization.

1 Introduction

The problem of designing electromagnetic cloaks can be described as follows (see, e.g., [20]). For a given incident wave, we design a scatterer in a region D_s such that an object in another region D_o is hidden from the incident wave. This means that the scatterer is designed such that the scattered wave cancels the incident wave in this region; in other words the amplitude of the superposition of the two fields is as small as possible. A 2D-scenario is sketched in Figure 1.

The problem has been cast and treated as a topology optimization problem in [9] and as a mixed-integer PDE-constrained optimization (MIPDECO) problem in [21]. We analyze and investigate the latter formulation. Specifically, we

* P. Manns and M. Winckler acknowledge funding by Deutsche Forschungsgemeinschaft through Priority Programme 1962, projects 15 and 22. This material was also supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, under Contract DE-AC02-06CH11357.

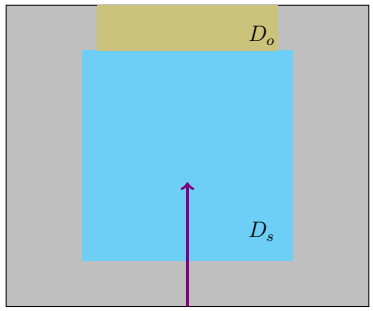


Fig. 1. Electromagnetic cloak problem: violet arrow indicates the angle of the incident wave, scatterer's region D_s is colored blue, and protected object's region D_o is colored olive.

optimize for a discrete-valued function, which assigns the material of the scatterer to each point in the region D_s such that it minimizes the superposition of the scattered electromagnetic field and the incident wave in the region D_o in a least-squares sense. We make use of analytical properties of the Helmholtz equation as in [9] and thus derive MIPDECO counterparts of the results from [9]. We employ the sum-up rounding (SUR) algorithm (see [8, 14, 18, 19]). Recently, SUR has been generalized to multidimensional problems in [13, 23].

We present an approximation of the electromagnetic field and the optimized objective value in the norm topology. Our method, however, suffers from the drawback that the design may exhibit chattering and therefore may not be implementable in reality. We demonstrate computationally that applying a median filtering heuristic as a postprocessing step can improve the implementability of the resulting control considerably while increasing the objective value moderately.

Notation For an optimization problem (P), we denote its feasible set by $\mathcal{F}_{(\mathrm{P})}$. For a domain D and $k \geq 1$, $H^k(D)$ denotes the Sobolev space of all functions in $L^2(D)$, that is square integrable functions, with distributional derivatives up to order k in $L^2(D)$. The space $L^2_{loc}(D)$ is the space of locally square integrable functions, that is functions that are square integrable over compact sets. Similarly, $H^k_{loc}(D)$ is the space of functions in $L^2_{loc}(D)$ with distributional derivatives up to order k in $L^2_{loc}(D)$. The Borel sigma algebra for a set $D \subset \mathbb{R}^d$ is denoted by $\mathcal{B}(D)$. The inner product of a Hilbert space H is denoted by $(\cdot,\cdot)_H$. We denote the compact embedding of a Hilbert space H into another Hilbert space K by $H \hookrightarrow^c K$. For a Banach space X, we denote convergence of a sequence $(x_n)_n \subset X$ to a limit $x \in X$ in the norm topology by $x_n \to x$, convergence in the weak topology by $x_n \rightharpoonup x$, and convergence in the weak* topology $x_n \rightharpoonup^* x$. For further background on concepts from PDE theory (see [17]). In the interest of a less distracting presentation, we will sometimes abbreviate $L^p(\mathbb{R}^d)$ by L^p, in particular for constant quantities.

2 State Equation

Here, we concentrate on 2D domains, and we use a simplified first-order absorbing boundary condition instead of the more complex Sommerfeld radiation condition. Extensions to 3D domains and the Sommerfeld radiation condition are analyzed in our companion paper [12].

We consider the state equation

$$-\Delta u - k_0^2(1+qw)u = k_0^2 qwu_0 \text{ in } D, \quad \frac{\partial u}{\partial n} - ik_0 u \quad = 0 \text{ on } \partial D, \tag{2.1}$$

where $w(s) \in [w_\ell, w_u]$ for a.a. $s \in D$ (see [9, Sect. 2]), which is analyzed in [5,9].

We state our assumptions on the regularity of the domain and the given data.

Assumption 1 *Let $D \subset \mathbb{R}^2$ be a bounded domain with a sufficiently smooth boundary such that the following assumptions hold:*

1. *The trace operator $\cdot|_{\partial D} : H^1(D) \to L^2(\partial D)$ exists and is continuous.*
2. *The scatterer $q \in L^\infty(D)$ satisfies $q \geq 0$ a.e. in D, and the incident field $u_0 \in H^2_{loc}(\mathbb{R}^d)$ solves the homogeneous equation $-\Delta u_0 - k_0^2 u_0 = 0$ on $\mathbb{R}^d$.*

Then, the existence and uniqueness of solutions follow from the results in [5].

Proposition 2 (Lem. 2.2,2.4 in [5]) *Let Assumption 1 hold. Then, for every $k_0 > 0$, the state equation (2.1) admits a unique weak solution in $H^1(D)$ for all $w \in \mathfrak{D}$. Moreover, for $k_0 > 0$ sufficiently small, the control-to-state operator $S : \mathfrak{D} \to H^1(D)$ is continuous.*

3 Optimal Design of Cloaks

Our aim is to design an electromagnetic scatterer to be effective in the region $D_o \subset D$. The scatterer is designed in the region $D_c \subset D \backslash D_o$, and we minimize the electromagnetic field in the region D_o, which corresponds to the superposition of the incidence wave and the response u, resulting in

$$\begin{aligned} &\inf_{u,w} \; J(u) := \frac{1}{2}\|u+u_0\|^2_{L^2(D_o)} \\ &\text{s.t. } -\Delta u - k_0^2(1+qw)u = k_0^2 qwu_0 \text{ in } D, \;\; \frac{\partial u}{\partial n} - ik_0 u = 0 \text{ on } \partial D, \\ &\qquad w(x) \in \{w_1, \ldots, w_M\} \text{ for a.a. } x \in D_c, \\ &\qquad w(x) = 0 \text{ for a.a. } x \in D \backslash D_c \end{aligned} \tag{P}$$

with discrete material constants $w_\ell = w_1 < \ldots < w_M = w_u$. We also define the continuous relaxation of (P) by relaxing the control to $w(x) \in [w_\ell, w_u]$. We derive a first-order optimality system for the relaxation problem denoted by (R). For the proofs, we refer again to [12]. Therefore, we analyze the Fréchet-differentiability of the control-to-state operator $S\colon L^\infty(D) \to H^1(D)$ and establish the well-posedness of the adjoint equation to obtain the desired optimality system.

Lemma 3 *Let Assumption 1 hold. Then, for $k_0 > 0$ sufficiently small, the operator $S\colon \mathfrak{D} \to H^1(D)$ is continuously Fréchet-differentiable and for every $w, h \in \mathfrak{D}$ its Fréchet-derivative $\hat{u} = S'(w)h \in H^1(D)$ is the unique function such that*

$$\int_D \nabla \hat{u} \cdot \nabla \overline{v} - k_0^2(1+qw)\hat{u}\overline{v}\,\mathrm{d}x - ik_0 \int_{\partial D} \hat{u}\overline{v}\,ds = k_0^2 \int_D qh(u+u_0)\overline{v}\,\mathrm{d}x \quad (3.1)$$

for all $v \in H^1(D)$, where $u = S(w)$ and $\overline{v}$ is the complex conjugate of v.

Lemma 4 *Under Assumption 1, we consider $k_0 > 0$ sufficiently small such that the assertions of Lemma 3 hold. Let $w, h \in \mathfrak{D}$ and $u = S(w) \in H^1(D)$. Then, there exists a unique $p \in H^1(D)$ such that the adjoint equation*

$$\int_D \nabla p \cdot \nabla \overline{v} - k_0^2(1+qw)p\overline{v}\,\mathrm{d}x - ik_0 \int_{\partial D} p\overline{v}\,ds = \int_{D_o} (\overline{u+u_0})\overline{v}\,\mathrm{d}x \quad (3.2)$$

holds for all $v \in H^1(D)$. Furthermore, the derivative f' with respect to w of the reduced objective $f(w) := J(S(w))$ satisfies $f'(w)h = k_0^2 \operatorname{Re}\left(\int_D qh(u+u_0)p\,\mathrm{d}x\right)$, where $\operatorname{Re}(\cdot)$ is the real part.

Theorem 5 *Let the assumptions from Lemma 4 be satisfied. If $(w^*, u^*) \in \mathcal{F}_{(\mathrm{R})}$ is a local solution of (R), then the solution of the adjoint equation $p^* \in H^1(D)$, established in Lemma 4, satisfies*

$$\operatorname{Re}\left(\int_D q(w-w^*)(u^*+u_0)p^*\,dx\right) \geq 0 \quad \forall w \in \mathfrak{D}.$$

4 Overall MIPDECO Algorithm

We begin this section by stating the algorithm to solve (P) approximately, which has been adopted from [13]. We then define its components. The main ingredient, namely the sum-up rounding (SUR) algorithm, is defined and analyzed in the next section.

Optimization Algorithm. The algorithm takes three inputs: an initial discretization $(\mathrm{R}_h^{(0)})$ of the continuous relaxation (R), an initial guess w_0^R for a local minimizer of $(\mathrm{R}_h^{(0)})$, and an initial rounding grid $T^{(0)}$, which is a set of cells that decompose the domain D. The algorithm proceeds iteratively. Each iteration n consists of five steps. First, the discretization of the continuous relaxation is refined, which yields the finite-dimensional relaxation $(\mathrm{R}_h^{(n)})$. We note that we do not require the discretization cells of the finite-dimensional NLP $(\mathrm{R}_h^{(n)})$ to coincide with those of the rounding grid – although this is possible and an intuitive choice. The subscript h denotes that a grid constant is associated with the approximation. Second, the NLP $(\mathrm{R}_h^{(n)})$ is solved to (local) optimality using the local minimizer of the previous iteration w_{n-1}^R as the initial guess. The resulting local minimizer is denoted by w_n^R. In the third step, the rounding grid is refined,

i.e. the cells are decomposed into smaller ones, which yields the rounding grid $T^{(n)}$. Fourth, the SUR algorithm is executed using the continuous control w_n^R and the rounding grid $T^{(n)}$ as inputs. It computes a discrete-valued control w_n^S, which is constant per cell of the rounding grid $T^{(n)}$. The fifth step computes a median-filtering w_n^F of w_n^S as a post-processing step.

Algorithm 1 Solving (P) approximately

Input: Initial guess w_0^R, approximation $(\mathrm{R}_h^{(0)})$ of (R), and rounding grid $T^{(0)}$.

for $n = 1, \dots$ **do**
  $(\mathrm{R}_h^{(n)}) \leftarrow$ refine approximation $(\mathrm{R}_h^{(n-1)})$ of (R)
  $w_n^R \leftarrow$ solve $(\mathrm{R}_h^{(n)})$ with initialization w_{n-1}^R
  $T^{(n)} \leftarrow$ refine $(T^{(n-1)})$
  $w_n^S \leftarrow \mathrm{SUR}(w_n^R, T^{(n)})$
  $w_n^F \leftarrow$ MEDIAN-FILTER(w_n^S)
end for

Refining and Solving the Continuous Relaxation. The first and second step of Algorithm 1 deal with the solution of (R). First, the discretization of the relaxation is refined, giving $(\mathrm{R}_h^{(n)})$. The discretization may for example be obtained by using piecewise affine globally continuous finite elements to discretize the state equation (2.1). Second, the finite-dimensional problem $(\mathrm{R}_h^{(n)})$ is solved to (local) optimality with a bound-constrained nonlinear optimization algorithm. We highlight that the problem is not convex as the control-to-state operator is not convex. Thus, we can only expect local optimality.

Our algorithm and its convergence results are agnostic to the specific choice of the discretization in the first and second step. However, under the assumptions on the rounding grid refinement that are introduced in Section 5, we obtain that for all weak* accumulation points $w^{R,*}$ of $(w_n^R)_n$ with approximating subsequences $w_{n_k}^R \rightharpoonup^* w^{R,*}$, we also obtain $w_{n_k}^S \rightharpoonup^* w^{R,*}$ with the help of the results from [13]. In addition, if an optimal consistency principle holds, i.e. if the sequence of local minimizers of the refined discretizations converges weakly* to a local minimizer of (R), see [1, 3, 10, 11, 15, 22], then the sequence $(w_n^S)_n$ is also optimal.

Median Filtering. The final step of Algorithm 1 consists of executing the algorithm MEDIAN-FILTERING. The algorithm takes a set of cells $S_1, \dots, S_N$ and a function w that is constant per cell, i.e. it can be represented as $w \in \mathbb{R}^N$, as inputs. The algorithm iterates through all cells of a rounding grid and computes the median value of all cells neighboring the current cell. Here, a cell is neighboring another one if the distance between them is below a certain threshold r. As we consider cells as sets, we need to define the distance between two sets S, $T \in \mathbb{R}^d$, which we do by means of the Pompeiu-Hausdorff distance, i.e. we define

$$d(S,T) := \max\left\{\sup_{s\in S}\inf_{t\in T}\|s-t\|_{\mathbb{R}^D}, \sup_{t\in T}\inf_{s\in S}\|s-t\|_{\mathbb{R}^D}\right\}.$$

We summarize the algorithm as Algorithm 2 below.

Algorithm 2 MEDIAN-FILTER

Input: Piecewise constant function w on a grid consisting of cells $S_1, \ldots, S_N$ with vector representation $w \in \mathbb{R}^N$, and radius r of the filter.
Output: Piecewise constant function w^F on the same grid on which w is defined in vector representation $w^F \in \mathbb{R}^N$

for $n = 1, \ldots, N$ **do**
  $w_i^F \leftarrow \arg\min_{w_j : d_{PH}(S_i, S_j) < r} |w_j - w_i|$
end for

We hope that applying a median-filtering heuristic reduces the switching without impacting the approximation severely. Unfortunately, it is straightforward to construct a counter example of a sequence of chattering functions with sum-up rounding that converges to a nonzero function such that the median at every grid cell evaluates to zero thereby destroying the approximation property.

5 Refinement and Multi-Dimensional Sum-Up Rounding

The refinement of the rounding grid is the third step in Algorithm 1. It produces a rounding grid $T^{(n)}$ from the previous $T^{(n-1)}$. We require a refinement that yields a weak* approximation of relaxed control by the discrete-valued control produced in the fourth step. This result depends on a suitable refinement and ordering of the grid cells, see also [13].

Definition 6 (Def. 4.9 in [13]) *We call a sequence* $\left(\left\{T_1^{(n)}, \ldots, T_{N^{(n)}}^{(n)}\right\}\right)_n \subset 2^{\mathcal{B}(D)}$ *of finite partitions of D an* order-conserving domain-dissection *of D if*

1. $N^{(0)} = 1$, $T_1^{(0)} = D$;
2. *for all n and for all $i \in \{1, \ldots, N^{(n-1)}\}$, there exist $1 \leq j < k \leq N^{(n)}$ such that $\bigcup_{\ell=j}^{k} T_\ell^{(n)} = T_i^{(n-1)}$, i.e. the order of the grid cells is preserved from $n-1$ to n;*
3. $\max_{i \in \{1, \ldots, N^{(n)}\}} \lambda(T_i^{(n)}) \to 0$, *where λ denotes the Lebesgue measure in $\mathbb{R}^d$, i.e. the maximum cell volume tends to zero, and;*
4. *the σ-algebra generated by $\bigcup_{n=1}^{\infty} \left\{T_1^{(n)}, \ldots, T_{N^{(n)}}^{(n)}\right\}$ is $\mathcal{B}(D)$, the Borel σ-algebra of the set D.*

These properties can be verified for the grids induced by the approximating sequences of space-filling curves, see [13], which induce a uniform refinement of the grid in every iteration, resulting in properties 1., 2. and 4. in Definition 6.

The fourth step of Algorithm 1 applies SUR to the control w_R^n using the rounding grid $T^{(n)}$. The algorithm makes use of a special ordered set of type 1 (SOS1) encoding of the discrete control realizations, see [19]. For a bounded

domain $D \subset \mathbb{R}^2$, we call a measurable function $\omega : D \to \{0,1\}^M$ such that $\sum_{i=1}^M \omega_i = 1$ a.e. in D holds a *binary control* and we call a measurable function $\alpha : D \to [0,1]^M$ such that $\sum_{i=1}^M \alpha_i = 1$ a.e. in D holds a *relaxed control*.

SUR computes binary controls ω, which are constant per grid cell from relaxed controls α. The relaxed control α can be obtained from a feasible continuous control w_n^R by solving $w_n^R(x) = \sum_{i=1}^M \alpha_i(x) w_i$ for a.a. $x \in D$. Once the binary control is computed, the discrete-valued control w_n^S, which is feasible for (P), can be reconstructed with $w_n^S(x) := \sum_{i=1}^M \omega_i(x) w_i$.

Algorithm 3 computes a binary control ω by iterating over the grid cells $1, \ldots, N$ and assigning a value to ω on the respective grid cell. It first computes the integrated signed difference between the relaxed control α and the resulting binary control ω over the grid cells, where ω is already defined. Next, the weighted mean of α over the current grid cell is added to this quantity. Third and last, ω is set to 1 on the grid cell where this sum has its maximum value and zero in the others. Ties are broken by choosing the smallest applicable index.

Algorithm 3 SUR (Multi-dimensional sum-up rounding)

Input: Rounding grid $T = \{S_1, \ldots, S_N\}$, and relaxed control α.
Output: Binary control ω.

for $i = 1, \ldots, N$ **do**

$\Phi_i := \int_{\bigcup_{\ell=1}^{i-1} S_\ell} \alpha(x) - \omega(x) \mathrm{d}x$

$\gamma_i := \Phi_i + \int_{S_i} \alpha(x) \mathrm{d}x$

$\tilde{\omega}_{i,j} = \begin{cases} 1 : j = \arg\max_{k \in \{1,\ldots,M\}} \gamma_{i,k}, \\ 0 : \text{else}, \end{cases}$ for all $j \in \{1, \ldots, M\}$

end for

return $\omega = \sum_{i=1}^N \chi_{S_i} \tilde{\omega}_i$, where χ_A denotes the indicator function of the set A.

The multi-dimensional SUR algorithm given in Algorithm 3 allows us to approximate a relaxed control, α, with a binary control, ω. Algorithm 3 does not generate an approximation in the norm topology, because the no binary control can approximate any relaxed control in norm. Instead, by executing SUR on a sequence of refined rounding grids, which constitute an order-conserving domain dissections we obtain weak* convergence of the corresponding sequence of ω^n to α in $L^\infty(D)$, denoted by $\omega_n \rightharpoonup^* \alpha$ in $L^\infty(D)$, i.e.

$$\int_D f(x)\omega_n(x)\mathrm{d}x \to \int_D f(x)\alpha(x)\mathrm{d}x \text{ for all } f \in L^1(D).$$

Proposition 7 *Let α be a relaxed control and $w_\alpha = \sum_{i=1}^M \alpha_i(s) w_i$ for $w \in \mathfrak{D}$. Then, the multi-dimensional SUR algorithm applied to a sequence of finite partitions of D that are an order conserving domain dissection produces a sequence of binary controls $(\omega_n)_n$ with corresponding $w_{\omega_n} = \sum_{i=1}^M (\omega_n)_i w_i$ such that*

1. *$\omega_n(s) \in \{0,1\}$ for all $s \in D$ and for all $n \in \mathbb{N}$,*

2. $\sup_{i\in\{1,\dots,N^{(n)}\}} \left\| \int_{\bigcup_{j=1}^{i} T_j^{(n)}} \alpha(s) - \omega_n(s)\mathrm{d}s \right\|_\infty \leq C \max_{i\in\{1,\dots,N^{(n)}\}} \lambda(S_i^{(n)})$ *for some* $C > 0$ *independent of the sequence of partitions, its indexing and* α,
3. $\omega_n \rightharpoonup^* \alpha$ *in* $L^\infty(D, \mathbb{R}^M)$,
4. $w_{\omega_n} \rightharpoonup^* w_\alpha$ *in* $L^\infty(D)$.

Proof. This follows immediately from [13, Theorem 4.7].

We summarize approximation relationship between (R) and (P) in Theorem 8, which follows from a more general result for the three-dimensional formulation with the Sommerfeld radiation condition that can be found in [12].

Theorem 8 *Let* $k_0 > 0$ *sufficiently small. Then, (R) admits a minimizer* $(u, w) \in H^1(D) \times \mathfrak{D}$. *Furthermore,*

$$\inf_{(u,w)\in\mathcal{F}_{(\mathrm{P})}} \frac{1}{2}\|u + u_0\|^2_{L^2(D_o)} = \min_{(u,w)\in\mathcal{F}_{(\mathrm{R})}} \frac{1}{2}\|u + u_0\|^2_{L^2(D_o)}.$$

We combine these results with [13] to obtain a convergence result for Algorithm 1. Theorem 9 shows that for a sequence of refined rounding grids and a sequence of controls that converge to a (local) minimizer of the continuous relaxation (R), the sequence of controls is integer in (P) and converges weakly* to the (local) minimizer of the continuous relaxation and approximates the (locally optimal) objective value arbitrarily close.

Theorem 9 *Let the assumptions of Proposition 2 hold. Let the sequence of rounding grids* $(T^{(n)})_n$ *be an order-conserving domain-dissection. Then, for every weak* accumulation point* $w^{R,*}$ *of the iterates* $(w_n^R)_n$ *with approximating iterates* $w_{n_k}^R \rightharpoonup^* w^{R,*}$ *in* $L^\infty(D)$ *and corresponding consistent sequence of relaxed controls* $\alpha_{n_k} \rightharpoonup^* \alpha^*$ *with* $\sum_{i=1}^M w_i(\alpha_{n_k})_i = w_{n_k}^R$ *and* $\sum_{i=1}^M w_i\alpha_i^* = w^{R,*}$ *produced by Algorithm 1, we obtain that*

$$J(S(w_{n_k}^S)) \to J(S(w^{R,*})).$$

Proof. Our results from the previous subsections and the assumption that the rounding grids constitute an order-conserving domain dissection enable us to apply [13, Thm 4.10] to our setting and the claim follows. □

SUR provides an efficient way to compute weak* approximations of relaxed controls. The resulting binary control is not optimal with respect to the number of switches for the provable bound, see [19], on the approximation. Optimizing with respect to switching cost while preserving the weak* approximation may require us to solve a different, more expensive, optimization problem, see [6].

6 Computational Results

We consider two discrete realizations for the material constant w in (P), namely $w_\ell = 0$ and $w_u = 1$. To solve the discretizations of the relaxed MIPDECO,

we use a limited-memory quasi-Newton method with BFGS updates for bound-constrained optimization.

We have obtained our numerical results on a laptop computer with INTEL(R) CORE(TM) I7-6820HQ CPU clocked at 2.70 GHz. We have used the open-source libraries FENICS [2] for the discretization of the state equation, DOLFIN-ADJOINT [7] for the computation of the reduced objective and its derivative by means of adjoint calculus, and PETSC [4] for the optimization of the reduced objective functional with the PETSC TAO solver [16].

The finest discretization grid we consider for the approximation of the state as well as the control vector is a uniform decomposition of the square domain into $2^9 \times 2^9$ square cells, which consist of two triangles each. We use piecewise affine, globally continuous Ansatz functions for the control of (R). For the integer-valued controls, i.e. the output of SUR, we have used piecewise constant (discontinuous Galerkin of order 0) Ansatz functions.

The main point of this paper is to show effectiveness of SUR, and we use the simpler Robin boundary conditions because we do not believe that using more involved boundary conditions adds significantly to our observations.

We set up an experiment with the parameter values $q = 0.75$ and $k_0 = 6\pi$ to validate our theoretical results. This corresponds to the setting in [9, 21]. Furthermore, the angle of the incident wave is set to $\pi/2$. We perform nine iterations of Algorithm 1, which requires about 12 hours on our laptop. For every $n \in \{1, \dots, 9\}$, we assess the convergence of the corresponding state vectors by inserting w_n^R and w_n^S into the numerical approximation of the control-to-state operator S on the n-th grid. Furthermore, w_n^R is projected onto the finest mesh and the self-convergence of the state vectors $\|S(w_n^R) - S(w_9^R)\|_{L^2}$ is reported. The results are summarized in Table 1 and validate Theorem 9. In particular, we observe that the difference between the state vector corresponding to the relaxed control and the binary control produced by SUR converges to zero. Furthermore, we observe self-convergence of the relaxed states $S(w_n^R)$. Interestingly, the objective $J(S(w_n^R))$ of the objective also appears to converge to zero, which hints at the possibility that cloaks may exist such that the electromagnetic field vanishes entirely in D_o. The real and imaginary parts of u_0, $S(w_9^R)$, $S(w_9^S)$ and $S(w_9^F)$ are plotted in Figure 2.

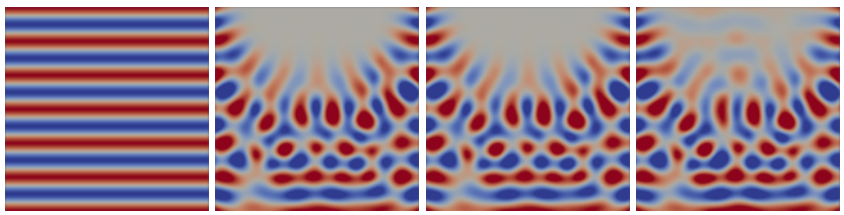
The effect of the filtering on the control can be observed in Figure 3. A lot of the scattered switching between material and no material placement is removed in the filtered control inputs in the bottom row. The objective is considerably higher, but with $J(w_9^F) = 3.671 \times 10^{-3}$, it is still small and the electromagnetic field is still damped heavily in the region D_o, see Figure 2.

7 Conclusion

We have successfully applied the SUR algorithm and the underlying approximation methodology to approximate the local infimum of the electromagnetic cloaking problem by taking the point of view of a non-convex MIPDECO. The computational results confirm our theoretical results.

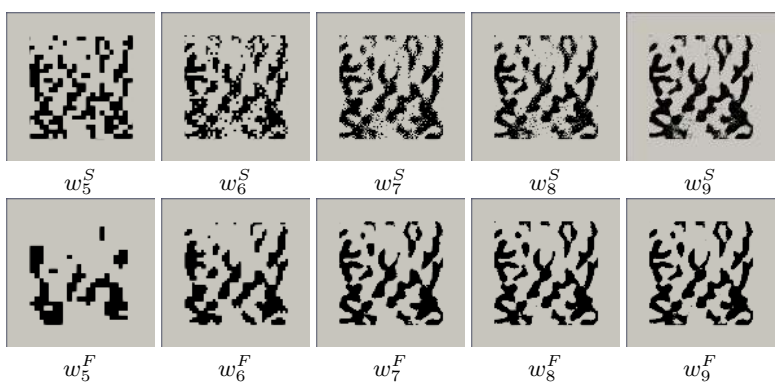
Table 1. State vector and objective convergence for refined grids.

n	# binary vars	h^n	$\|S(w_n^S) - S(w_n^R)\|$	$\|S(w_n^R) - S(w_9^R)\|$	$J(S(w_n^R))$
3	32	3.54×10^{-1}	1.752×10^{-1}	1.795×10^{0}	2.263×10^{-1}
4	128	0.18×10^{-1}	3.062×10^{-1}	1.939×10^{0}	1.991×10^{-1}
5	504	8.84×10^{-2}	2.400×10^{-1}	2.022×10^{0}	1.624×10^{-1}
6	2008	4.42×10^{-2}	2.151×10^{-1}	9.538×10^{-1}	2.188×10^{-2}
7	8028	2.21×10^{-2}	5.386×10^{-2}	2.136×10^{-1}	1.999×10^{-3}
8	32112	1.10×10^{-2}	2.096×10^{-2}	3.947×10^{-2}	9.852×10^{-5}
9	128452	5.52×10^{-3}	4.125×10^{-3}	0	3.533×10^{-6}



$\mathrm{Re}(S(0) + u_0)$ $\mathrm{Re}(S(w_9^R) + u_0)$ $\mathrm{Re}(S(w_9^S) + u_0)$ $\mathrm{Re}(S(w_9^F) + u_0)$

Fig. 2. Real part of the incident wave u_0, the optimally scattered wave for (R) and the scattered waves due to SUR and the median filtering.



w_5^S w_6^S w_7^S w_8^S w_9^S

w_5^F w_6^F w_7^F w_8^F w_9^F

Fig. 3. Control vector iterates (top) and their filterings (bottom).

Acknowledgments

The authors thank Dirk Lorenz, TU Braunschweig, for pointing at median filterings for reduction of chattering.

References

1. Ali, A.A., Deckelnick, K., Hinze, M.: Error analysis for global minima of semilinear optimal control problems. arXiv preprint arXiv:1705.01201 (2017)
2. Alnæs, M.S., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M.E., Wells, G.N.: The FEniCS Project Version 1.5. Archive of Numerical Software **3**(100) (2015), https://doi.org/10.11588/ans.2015.100.20553
3. Arada, N., Casas, E., Tröltzsch, F.: Error estimates for the numerical approximation of a semilinear elliptic control problem. Computational Optimization and Applications **23**(2), 201–229 (Nov 2002). https://doi.org/10.1023/A:1020576801966, https://doi.org/10.1023/A:1020576801966
4. Balay, S., Abhyankar, S., Adams, M.F., B., J., Brune, P., Buschelman, K., Dalcin, L., Eijkhout, V., Kaushik, D., Knepley, M.G., McInnes, L.C., Gropp, W.D., Rupp, K., Smith, B.F., Zampini, S., Zhang, H.: PETSc Users Manual. Tech. Rep. ANL-95/11 - Revision 3.7, Argonne National Laboratory (2016)
5. Bao, G., Li, P.: Inverse medium scattering for the Helmholtz equation at fixed frequency. Inverse Problems **21**(5), 1621 (2005)
6. Bestehorn, F., Hansknecht, C., Kirches, C., Manns, P.: A switching cost aware rounding method for relaxations of mixed-integer optimal control problems. In: Proceedings of the IEEE Conference on Decision and Control (2019), to appear
7. Farrell, P.E., Ham, D.A., Funke, S.W., Rognes, M.E.: Automated derivation of the adjoint of high-level transient finite element programs. SIAM Journal on Scientific Computing **35**(4), C369–C393 (2013)
8. Hante, F.M., Sager, S.: Relaxation methods for mixed-integer optimal control of partial differential equations. Computational Optimization and Applications **55**(1), 197–225 (2013), https://doi.org/10.1007/s10589-012-9518-3
9. Haslinger, J., Mäkinen, R.A.E.: On a topology optimization problem governed by two-dimensional Helmholtz equation. Computational Optimization and Applications **62**(2), 517–544 (2015), https://doi.org/10.1007/s10589-015-9746-4
10. Hinze, M.: A variational discretization concept in control constrained optimization: the linear-quadratic case. Computational Optimization and Applications **30**(1), 45–61 (2005), https://doi.org/10.1007/s10589-005-4559-5
11. Kröner, A., Vexler, B.: A priori error estimates for elliptic optimal control problems with a bilinear state equation. Journal of Computational and Applied Mathematics **230**(2), 781 – 802 (2009). https://doi.org/https://doi.org/10.1016/j.cam.2009.01.023, http://www.sciencedirect.com/science/article/pii/S0377042709000260
12. Leyffer, S., Manns, P., Winckler, M.: Convergence of sum-up rounding schemes for the electromagnetic cloak problem. to be submitted
13. Manns, P., Kirches, C.: Multi-dimensional sum-up rounding for elliptic control systems. Submitted (2018), https://spp1962.wias-berlin.de/preprints/080r.pdf
14. Manns, P., Kirches, C.: Improved regularity assumptions for partial outer convexification of mixed-integer PDE-constrained optimization problems. ESAIM: Control, Optimisation and Calculus of Variations (2019), to appear, https://doi.org/10.1051/cocv/2019016
15. Meyer, C., Rösch, A.: Superconvergence properties of optimal control problems. SIAM Journal on Control and Optimization **43**(3), 970–985 (2004). https://doi.org/10.1137/S0363012903431608, https://doi.org/10.1137/S0363012903431608

16. Munson, T., Sarich, J., Wild, S., Benson, S., McInnes, L.C.: TAO 3.5 Users Manual. Tech. Rep. ANL/MCS-TM-322, Argonne National Laboratory, Mathematics and Computer Science Division (2015)
17. Renardy, M., Rogers, R.C.: An Introduction to Partial Differential Equations, vol. 13. Springer Science & Business Media (2006), https://doi.org/10.1007/b97427
18. Sager, S.: Numerical methods for mixed-integer optimal control problems. Der andere Verlag, Tönning, Lübeck, Marburg (2005)
19. Sager, S., Bock, H.G., Diehl, M.: The integer approximation error in mixed-integer optimal control. Mathematical Programming, Series A **133**(1–2), 1–23 (2012), https://doi.org/10.1007/s10107-010-0405-3
20. Schurig, D., Mock, J.J., Justice, B.J., Cummer, S.A., Pendry, J.B., Starr, A.F., Smith, D.R.: Metamaterial electromagnetic cloak at microwave frequencies. Science **314**(5801), 977–980 (2006)
21. Vogt, R.H., Leyffer, S., Munson, T.: A mixed-integer PDE-constrained formulation for electromagnetic cloaking. Submitted (2019)
22. Yousept, I.: Finite element analysis of an optimal control problem in the coefficients of time-harmonic eddy current equations. Journal of Optimization Theory and Applications **154**(3), 879–903 (Sept 2012). https://doi.org/10.1007/s10957-012-0040-7, https://doi.org/10.1007/s10957-012-0040-7
23. Yu, J., Anitescu, M.: Multidimensional sum-up rounding for integer programming in optimal experimental design. Mathematical Programming pp. 1–40 (2019)