SUPPLEMENTARY MATERIALS: A METHOD FOR CONVEX BLACK-BOX INTEGER GLOBAL OPTIMIZATION*

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SM1. Benchmark problems. Table SM1 shows the set of benchmark problems considered in our paper.

Table SM1: Set of convex benchmark problems.

Name	Expression	$f(x^*)$	x^*
`CB3II` [?]	$\max\left\{\sum_{i=1}^{n-1} x_i^4 + x_{i+1}^2, \sum_{i=1}^{n-1} (2-x_i)^2 + (2-x_{i+1})^2, \sum_{i=1}^{n-1} 2e^{-x_i+x_{i+1}}\right\}$	$2(n-1)$	e
`CB3I` [?]	$\sum_{i=1}^{n-1} \max\left\{x_i^4 + x_{i+1}^2, (2-x_i)^2 + (2-x_{i+1})^2, 2e^{-x_i+x_{i+1}}\right\}$	$2(n-1)$	e
`KLT` [?]	$\max_{i\in\{1,\dots,n\}} \left\{\lVert x - c_i - 2e\rVert^2\right\}, c_i = 2e_i - e$	n	$2e$
`LQ` [?]	$\sum_{i=1}^{n-1} \max\left\{-x_i - x_{i+1}, -x_i - x_{i+1} + x_i^2 + x_{i+1}^2 - 1\right\}$	$-(n-1)$	many
`abhi` [?]	$\sum_{i=1}^{n} \left[64\left(c_1(x_i-2) - c_2(x_{i+1}-2)\right)^2 + \left(c_2(x_i-2) - c_1(x_{i+1}-2)\right)^2\right],$ $c_1 = \cos\left(\frac{\pi}{8}\right), c_2 = \sin\left(\frac{\pi}{8}\right)$	0	$2e$
`maxq` [?]	$\max_{i\in\{1,\dots,n\}} \left\{x_i^2\right\}$	0	0
`mxhilb` [?]	$\max_{i\in\{1,\dots,n\}} \left\{\sum_{j=1}^{n} \left\lvert \frac{x_j}{i+j-1} \right\rvert\right\}$	0	0
`quad`	$\sum_{i=1}^{n} (x_i - 2)^2$	0	$2e$

SM2. Performance of MILP model. Table SM2 shows the size of the MILP model at each iteration and the computational effort required for solving it. The column k refers to the iteration of Algorithm 1, *sHyp* denotes the number of secants (i.e., $\lvert W(X)\rvert$), and *LB* and *UB* give the lower and upper bound on f on Ω, respectively. We show the computational effort needed to solve each MILP via *time*, the mean solution time (in seconds) for 5 replications; *simIter*, the number of simplex iterations; and *nodes*, the number of branch-and-bound nodes explored by the MILP solver. The size of each MILP (after presolve) is shown in terms of *bVars*, the number of binary variables; *cVars*, the number of continuous variables; and *cons*, the number of constraints. Table SM2 also

*Submitted to the editors DATE.
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Table SM2: Characteristics of the first 12 instances of (CPF) generated by Algorithm 1 minimizing the convex quadratic function abhi on $\Omega = [-2, 2]^3 \cap \mathbb{Z}^3$. (CPF) instances are generated by AMPL and solved by CPLEX; times are the mean of five replications.

k	*sHyp*	*LB*	*UB*	*time*	*simIter*	*nodes*	*bVars*	*cVars*	*cons*	$\hat{x}$
1	20	-616.3	79.9	0.1	374	0	335	268	960	$[2; 2; -2]$
2	52	-555.1	79.9	3.9	13,180	8,089	847	685	2,466	$[2; 2; -1]$
3	100	-475.2	44.7	10.9	31,423	8,555	1,615	1,310	4,724	$[2; 1; -2]$
4	172	-434.4	44.7	7.3	19,267	1,728	2,767	2,247	8,110	$[1; 2; -2]$
5	276	-413.9	19.1	30.8	68,264	5,874	4,431	3,600	13,000	$[2; 1; -1]$
6	418	-373.1	19.1	95.1	84,031	7,933	6,703	5,447	19,676	$[1; 2; -1]$
7	611	-311.7	19.1	59.6	83,102	5,440	9,791	7,957	28,749	$[2; -2; -2]$
8	866	-293.2	19.1	99.6	86,318	3,933	13,871	11,273	40,736	$[1; 1; -2]$
9	1,196	-232.0	19.1	154.2	84,440	4,568	19,151	15,564	56,248	$[1; 1; -1]$
10	1,532	-199.5	19.1	452.3	235,473	6,400	24,527	19,933	72,042	$[2; -2; -1]$
11	2,038	-192.9	19.1	1,006	387,491	9,686	32,623	26,512	95,826	$[2; -1; -2]$
12	2,605	-140.9	19.1	964.3	455,939	29,279	41,695	33,884	122,477	$[1; -1; -2]$

shows the optimal solution $\hat{x}$ of each MILP. These experiments were performed by using CPLEX (v.12.6.1.0) on a 2.20 GHz, 12-core Intel Xeon computer with 64 GB of RAM. For this small problem we see that the size of the MILP grows exponentially as the iterations proceed, which results in an exponential growth in solution time as illustrated in Figure 4. The iteration 13 MILP was not solved after 30 minutes.

SM3. Detailed Numerical Results. Tables SM3–SM8 contain detailed numerical results for the interested reader. Note that some solvers do not respect the given budget of function evaluations. We have used a different stopping criterion for MATSuMoTo: it is set to stop only when a point with the optimal value has been identified. Also, although the global minimum is the starting point for maxq and mxhilb, MATSuMoTo instead uses its initial symmetric Latin hypercube design. The last row of Table 1 in Section 2 shows $|\Omega|$ for these problems.

Table SM3: Number of function evaluations before solvers terminate for $n = 3$ test problems. Parentheses show number of evaluations until x^* is evaluated.

	SUCIL-ideal1	SUCIL	DFLINT	DFLINT-M	NOMAD	NOMAD-NM	MATSuMoTo
abhi	19 (8)	30 (17)	161 (57)	150 (10)	59 (20)	129 (56)	60 (31)
quad	25 (8)	39 (17)	157 (54)	150 (10)	53 (18)	119 (35)	45 (16)
KLT	22 (8)	28 (13)	152 (48)	146 (15)	51 (24)	116 (34)	46 (17)
maxq	14 (1)	14 (1)	181 (1)	181 (1)	34 (1)	107 (1)	50 (21)
mxhilb	16 (1)	21 (1)	181 (1)	181 (1)	35 (1)	106 (1)	48 (19)
LQ	18 (8)	36 (7)	182 (7)	181 (7)	48 (7)	107 (7)	41 (12)
CB3I	22 (8)	25 (10)	184 (22)	181 (7)	56 (12)	108 (12)	60 (31)
CB3II	22 (8)	34 (11)	184 (22)	181 (7)	44 (12)	108 (12)	60 (31)

Table SM4: Number of function evaluations before solvers terminate for $n = 4$ test problems. Parentheses show number of evaluations until x^* is evaluated.

	SUCIL-ideal1	SUCIL	DFLINT	DFLINT-M	NOMAD	NOMAD-NM	MATSuMoTo
abhi	45 (10)	75 (41)	993 (137)	959 (13)	110 (16)	453 (88)	89 (60)
quad	51 (10)	95 (21)	982 (124)	959 (13)	111 (12)	460 (89)	56 (25)
KLT	51 (10)	67 (21)	954 (141)	953 (20)	116 (42)	457 (55)	54 (23)
maxq	30 (1)	33 (1)	1,001 (1)	1,001 (1)	91 (1)	451 (1)	89 (60)
mxhilb	32 (1)	65 (1)	1,001 (1)	1,001 (1)	90 (1)	450 (1)	63 (34)
LQ	39 (10)	109 (15)	1,000 (17)	1,001 (9)	145 (9)	453 (10)	47 (18)
CB3I	50 (10)	58 (14)	1,001 (58)	1,001 (9)	149 (42)	456 (23)	126 (81)
CB3II	48 (10)	91 (14)	1,001 (50)	1,001 (19)	125 (30)	460 (35)	131 (76)

Table SM5: Number of function evaluations before solvers terminate for $n = 5$ test problems. Parentheses show number of evaluations until x^* is evaluated.

	SUCIL-ideal1	SUCIL	DFLINT	DFLINT-M	NOMAD	NOMAD-NM	MATSuMoTo
abhi	105 (12)	154 (113)	1,001 (167)	1,000 (16)	301 (41)	1,000 (156)	214 (157)
quad	108 (12)	146 (58)	1,001 (186)	1,000 (16)	286 (26)	1,000 (58)	113 (62)
KLT	108 (12)	121 (69)	1,001 (193)	1,001 (25)	296 (43)	1,000 (176)	108 (77)
maxq	75 (1)	80 (1)	1,001 (1)	1,001 (1)	257 (1)	1,000 (1)	284 (255)
mxhilb	96 (1)	154 (1)	1,001 (1)	1,001 (1)	257 (1)	1,000 (1)	131 (102)
LQ	83 (12)	126 (17)	1,001 (44)	1,001 (11)	425 (15)	1,000 (15)	56 (27)
CB3I	114 (12)	155 (68)	1,001 (103)	1,001 (11)	417 (18)	1,000 (18)	266 (237)
CB3II	100 (12)	135 (66)	1,001 (130)	1,001 (26)	465 (69)	1,000 (54)	281 (224)

Table SM6: Number of function evaluations taken by 20 replications of MATSuMoTo for each of the 8 convex test problems for $n = 3$.

	abhi	quad	KLT	maxq	mxhilb	LQ	CB3I	CB3II
1	48 (19)	48 (19)	48 (19)	43 (14)	48 (19)	43 (14)	59 (30)	83 (54)
2	63 (34)	44 (15)	44 (15)	48 (19)	44 (15)	43 (14)	43 (14)	54 (25)
3	44 (15)	43 (14)	45 (16)	48 (19)	49 (20)	43 (14)	48 (19)	106(77)
4	58 (29)	43 (14)	48 (19)	53 (24)	64 (35)	39 (10)	48 (19)	63 (34)
5	79 (50)	48 (19)	49 (20)	54 (25)	44 (15)	38 (9)	53 (24)	53 (24)
6	73 (44)	37 (9)	48 (19)	39 (10)	53 (24)	39 (10)	64 (35)	53 (24)
7	88 (59)	43 (14)	44 (15)	58 (29)	49 (20)	44 (15)	54 (25)	59 (30)
8	60 (31)	48 (19)	43 (14)	38 (9)	43 (14)	44 (15)	73 (44)	73 (44)
9	79 (50)	43 (14)	48 (19)	53 (24)	43 (14)	39 (10)	68 (39)	58 (29)
10	65 (36)	53 (24)	49 (20)	53 (24)	49 (20)	44 (15)	49 (20)	43 (14)
11	63 (34)	48 (19)	48 (19)	48 (19)	58 (29)	43 (14)	63 (34)	48 (19)
12	53 (24)	44 (15)	43 (14)	48 (19)	48 (19)	38 (9)	58 (29)	53 (24)
13	48 (19)	49 (20)	48 (19)	54 (25)	49 (20)	53 (24)	68 (39)	37 (9)
14	43 (14)	48 (19)	48 (19)	43 (14)	49 (20)	43 (14)	78 (49)	49 (20)
15	44 (15)	48 (19)	48 (19)	48 (19)	48 (19)	43 (14)	39 (10)	73 (44)
16	58 (29)	48 (19)	43 (14)	58 (29)	37 (9)	38 (9)	58 (29)	73 (44)
17	48 (19)	44 (15)	43 (14)	53 (24)	49 (20)	37 (9)	68 (39)	48 (19)
18	64 (35)	43 (14)	48 (19)	63 (34)	53 (24)	38 (9)	93 (64)	69 (40)
19	74 (45)	43 (14)	48 (19)	58 (29)	53 (24)	37 (9)	68 (39)	63 (34)
20	49 (20)	43 (14)	43 (14)	58 (29)	48 (19)	44 (15)	54 (25)	58 (29)
$\lfloor$mean$\rfloor$	**60 (31)**	**45 (16)**	**46 (17)**	**50 (21)**	**48 (19)**	**41 (12)**	**60 (31)**	**60 (31)**

Table SM7: Number of function evaluations taken by 20 replications of MATSuMoTo for each of the 8 convex test problems for $n = 4$.

	abhi	quad	KLT	maxq	mxhilb	LQ	CB3I	CB3II
1	110 (81)	56 (27)	51 (22)	66 (37)	60 (31)	41 (12)	230 (20)	85 (56)
2	70 (41)	55 (26)	60 (31)	120 (91)	60 (31)	46 (17)	76 (47)	85 (56)
3	50 (21)	50 (21)	50 (21)	94 (65)	50 (21)	41 (12)	150 (121)	196 (11)
4	104 (75)	50 (21)	50 (21)	65 (36)	80 (51)	45 (16)	110 (81)	186 (16)
5	91 (62)	60 (11)	55 (26)	112 (83)	80 (51)	41 (12)	100 (11)	161 (132)
6	66 (37)	55 (26)	56 (27)	90 (61)	56 (27)	55 (26)	101 (72)	196 (167)
7	86 (57)	65 (36)	45 (16)	65 (36)	50 (21)	46 (17)	90 (61)	141 (112)
8	90 (61)	55 (26)	65 (11)	81 (52)	65 (36)	41 (12)	105 (63)	346 (161)
9	166 (137)	56 (27)	56 (27)	55 (26)	50 (21)	51 (22)	96 (67)	100 (71)
10	85 (56)	65 (21)	55 (11)	60 (31)	65 (36)	55 (26)	55 (26)	106 (77)
11	147 (118)	70 (41)	46 (17)	70 (41)	65 (36)	46 (17)	95 (66)	80 (51)
12	96 (67)	45 (16)	46 (17)	112 (83)	71 (42)	50 (21)	112 (83)	231 (202)
13	39 (11)	60 (31)	71 (42)	119 (90)	71 (42)	41 (12)	50 (21)	81 (52)
14	85 (56)	61 (32)	50 (21)	85 (56)	55 (26)	55 (26)	205 (176)	55 (26)
15	101 (72)	60 (31)	60 (31)	94 (65)	55 (26)	41 (12)	160 (131)	114 (85)
16	66 (37)	51 (22)	45 (16)	50 (21)	75 (46)	41 (12)	65 (36)	65 (36)
17	76 (47)	65 (36)	56 (27)	55 (26)	60 (31)	60 (31)	85 (56)	131 (102)
18	70 (41)	56 (27)	51 (22)	110 (81)	75 (46)	65 (36)	314 (285)	70 (41)
19	81 (52)	56 (11)	60 (31)	221 (192)	71 (42)	41 (12)	117 (16)	141 (52)
20	105 (76)	46 (17)	60 (31)	66 (37)	50 (21)	56 (27)	210 (13)	50 (21)
$\lfloor \text{mean} \rfloor$	**89 (60)**	**56 (25)**	**54 (23)**	**89 (60)**	**63 (34)**	**47 (18)**	**126 (81)**	**131 (76)**

Table SM8: Number of function evaluations taken by 20 replications of MATSuMoTo for each of the 8 convex test problems for $n = 5$.

	abhi	quad	KLT	maxq	mxhilb	LQ	CB3I	CB3II
1	472 (443)	82 (53)	83 (54)	137 (108)	149 (120)	52 (23)	199 (170)	83 (54)
2	358 (329)	83 (54)	317 (288)	229 (200)	208 (179)	63 (34)	267 (238)	518 (489)
3	193 (164)	83 (54)	172 (143)	115 (86)	175 (146)	62 (33)	377 (348)	1,626 (133)
4	194 (165)	73 (44)	62 (33)	171 (142)	72 (43)	57 (28)	223 (194)	325 (296)
5	160 (131)	72 (43)	73 (44)	504 (475)	87 (58)	53 (24)	835 (806)	487 (458)
6	396 (367)	187 (127)	78 (49)	92 (63)	152 (123)	53 (24)	127 (98)	196 (18)
7	82 (53)	82 (53)	120 (91)	135 (106)	247 (218)	62 (33)	184 (155)	603 (574)
8	62 (33)	106 (15)	58 (29)	969 (940)	186 (157)	52 (23)	164 (135)	67 (38)
9	107 (78)	83 (54)	68 (39)	276 (247)	103 (74)	57 (28)	399 (370)	671 (642)
10	186 (13)	78 (18)	73 (44)	352 (323)	82 (53)	73 (44)	364 (335)	205 (18)
11	181 (152)	130 (65)	62 (33)	255 (226)	53 (24)	58 (29)	414 (385)	240 (211)
12	239 (210)	115 (86)	125 (96)	403 (374)	181 (152)	47 (18)	83 (54)	462 (433)
13	295 (211)	87 (18)	72 (43)	167 (138)	151 (122)	48 (19)	128 (99)	324 (33)
14	354 (325)	78 (49)	72 (43)	236 (207)	82 (53)	62 (33)	200 (171)	154 (125)
15	113 (13)	73 (44)	52 (14)	512 (483)	88 (59)	53 (24)	271 (242)	87 (58)
16	227 (23)	82 (53)	120 (91)	189 (160)	93 (64)	68 (39)	326 (297)	63 (34)
17	102 (73)	135 (106)	270 (241)	195 (166)	57 (28)	58 (29)	180 (151)	599 (570)
18	139 (110)	268 (80)	150 (121)	173 (144)	112 (83)	58 (29)	228 (199)	112 (83)
19	169 (19)	67 (38)	72 (43)	108 (79)	117 (88)	42 (13)	204 (175)	93 (64)
20	268 (239)	301 (195)	62 (13)	465 (436)	237 (208)	53 (24)	162 (133)	180 (151)
$\lfloor \text{mean} \rfloor$	**214 (157)**	**113 (62)**	**108 (77)**	**284 (255)**	**131 (102)**	**56 (27)**	**266 (237)**	**281 (224)**