A Globally Convergent Filter Method for MPECs

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Abstract

We propose a new method for mathematical programs with complementarity constraints that is globally convergent to B-stationary points. The method solves a linear program with complementarity constraints to obtain an estimate of the active set. It then fixes the activities and solves an equality-constrained quadratic program to obtain fast convergence. The method uses a filter to promote global convergence. We establish convergence to B-stationary points.

Keywords: Mathematical Programs with Equilibrium Constraints, Mathematical Programs with Complementarity Constraints, B-Stationarity, Sequential Linear Programming.

AMS-MSC2000: 90C33, 90C55

1   Introduction

Mathematical programs with equilibrium constraints (MPECs) arise is a wide variety of applications (Ferris and Pang, 1997; Luo et al., 1996; Pang and Leyffer, 2004), as is evident from the rich set of test problems (Leyffer, 2000; Dirkse, 2001). MPECs are conveniently expressed as

$$
\begin{aligned}
\underset{x,y}{\text{minimize}} \quad & f(x,y) \\
\text{subject to} \quad & c(x,y) \geq 0 \\
& 0 \leq y \ \perp \ F(x,y) \geq 0,
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, $p + q = n$, and f, c, and F are smooth functions from $\mathbb{R}^n$ to $\mathbb{R}, \mathbb{R}^m$, and $\mathbb{R}^q$, respectively. More general constraints are readily included in (1.1). For convenience, we also abbreviate the variables as $z = (x, y)$.

Recently, many authors have suggested nonlinear programming (NLP) methods for solving MPECs, (Anitescu, 2005; Benson et al., 2006; Friedlander et al., 2005; Fletcher and Leyffer, 2004; Fletcher et al., 2006; Leyffer, 2003, 2006; Leyffer et al., 2006; Liu et al., 2005; Raghunathan and Biegler, 2005). These methods can be very successful and have enabled us to solve much larger problems than previously possible. However, the NLP approach does not preclude convergence to spurious stationary points. We have observed failures of the NLP approach on some large and difficult problems arising in electricity markets (Chen et al., 2006) that we believe are a manifestation of fundamental shortcomings of the NLP approach. The aim of this paper is to present a new method that avoids these shortcomings.

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Our algorithm is motivated by Bouligand stationarity, or B-stationarity; defined as follows, see, for example (Scholtes, 2002).

Definition 1.1. *A point (x^*, y^*) is called* Bouligand stationary, *or* B-stationary, *if $d = 0$ solves the linear program with equilibrium constraints (LPEC) obtained by linearizing f, c, and F about (x^*, y^*),*

$$
\begin{aligned}
\underset{d}{\text{minimize}} \quad & {g^*}^T d \\
\text{subject to} \quad & c^* + {A^*}^T d \geq 0, \\
& 0 \leq y^* + d_y \ \perp \ F^* + {B^*}^T d \geq 0,
\end{aligned}
$$

where $g^* = \nabla f(x^*, y^*)$, $A^* = \nabla c(x^*, y^*)$, $B^* = \nabla F(x^*, y^*)$, and $d = (d_x, d_y)$ is a partition of the step into its x- and y-components.

Our new method is related to recent sequential linear programming (SLP) methods that involve a second-order equality-constrained quadratic programming (EQP) step. Such methods were originally proposed by Fletcher and de la Maza (1989), and have recently received renewed interest. Chin and Fletcher (2003) propose a filter to enforce global convergence, and Byrd et al. (2004) consider a penalty function approach. The latter approach has also been implemented as SLIQUE. The key idea of these methods is to solve a linear program at each iteration to predict the optimal active set. Given this prediction of the active set, an EQP is solved to ensure fast convergence. These methods are computationally efficient, because there exist efficient implementations for the solution of both subproblems (LP and EQP). SLP-EQP methods can be regarded as a computationally efficient implementation of sequential quadratic programming methods.

Our main contribution is to regard the complementarity constraint as as a *structural constraint*, not as a nonlinear equation. We believe this is an important ingredient in the derivation of robust methods for MPECs. We extend popular SLP-EQP methods to MPECs. At each iteration of our method, we solve an LPEC inside a trust region. The solution of the LPEC provides a first-order step and an estimate of the optimal active set. We fix the active constraints predicted by the LPEC and then solve an EQP to ensure fast convergence. Global convergence is promoted through the use of a three-dimensional filter that splits the constraint violation into the complementarity constraints and the components corresponding to the general constraints.

The remainder of this paper is organized as follows. In Section 2 we review stationarity concepts and discuss their respective weakness by providing small examples. In Section 3 we define our new filter method for solving MPECs, and in Section 4 we establish convergence to B-stationary points. In Section 5 we describe how the method can be accelerated by adding an EQP phase.

Notation. We use subscripts to identify components of vectors, or matrices, and superscripts $z^{(k)}$ to indicate iterates. Similarly, functions that are evaluated at particular points are denoted as $f^{(k)} := f(z^{(k)})$ and so forth.

2   Motivation

The past five years have seen exciting algorithmic developments in MPECs. In Particular, a range of new stationarity concepts has been developed. Coinciding with these developments, NLP methods have been shown to converge to certain stationary points. Upon closer inspection, however, most of the new stationarity conditions turn out not preclude the existence of first-order descend directions, making their practical value questionable.

Even worse, NLP solvers do sometimes converge to such spurious stationary points. We have observed this behavior in an application involving the Pennsylvania-Jersey-Maryland electricity market (Chen et al., 2006). Surprisingly, the recent SQPEC method by Scholtes (2004) can also converge to spurious stationary points, despite the fact that it preserves the complementarity condition as a structural constraint.

2.1 The Alphabet Soup of MPEC Stationarity

Many stationarity concepts have been proposed for MPECs; see Scheel and Scholtes (2000). Here, we argue that all but one of these definitions are misleading, because they do not preclude the existence of descend directions.

MPEC stationarity is defined in terms of the following first-order conditions.

Definition 2.1. *A point $z^* = (x^*, y^*)$ is called weakly stationary if there exist multipliers λ, μ, and ν such that*

$$g^* - A^{*T}\lambda - B^{*T}\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} = 0,$$

$$0 \leq c^* \perp \lambda \geq 0, \tag{2.1}$$

$$0 \leq y^* \perp F^* \geq 0,$$

$$y_i^* > 0 \implies \mu_i = 0, \quad and \quad F_j^* > 0 \implies \nu_j = 0, \ \forall j = 1, \ldots, q.$$

This weakest form of "stationarity" is tightened by considering the index set of degenerate complementarity conditions:

$$\mathcal{D}(z) := \big\{ i : y_i = 0 = F_i(z) \big\} \tag{2.2}$$

Clearly, weakly stationary points allow trivial descend directions if $\mu_i < 0$ or $\nu_i < 0$ for some $i \in \mathcal{D}^*$. Unfortunately, only strong stationarity precludes the existence of such descend directions. All other stationarity concepts allow trivial descend directions and are, in our view, useless.

Definition 2.2. *Let $z^* = (x^*, y^*)$, and $\mathcal{D}^* := \mathcal{D}(z^*)$ be the set of degenerate indices.*

1. *(x^*, y^*) satisfying (2.1) is called* strongly stationary *if*

$$\mu_i \geq 0, \quad and \quad \nu_i \geq 0, \ \forall i \in \mathcal{D}^*. \tag{2.3}$$

2. *(x^*, y^*) satisfying (2.1) is called* A-stationary *if*

$$\mu_i \geq 0 \ \ or \ \ \nu_i \geq 0, \ \forall i \in \mathcal{D}^*. \tag{2.4}$$

3. *(x^*, y^*) satisfying (2.1) is called* C-stationary *if*

$$\mu_i \nu_i \geq 0 \ \forall i \in \mathcal{D}^*. \tag{2.5}$$

4. *(x^*, y^*) satisfying (2.1) is called* M-stationary *if*

$$\big(\mu_i > 0 \ \ and \ \ \nu_i > 0\big) \ or \ \mu_i \nu_i = 0, \ \forall i \in \mathcal{D}^*. \tag{2.6}$$
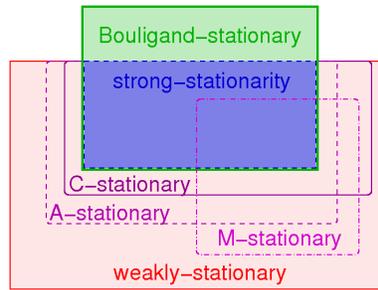
Figure 1: Relationships among MPEC stationarity concepts.

We visualize the relationships among these (confusing) stationarity concepts in Figure 1. Scheel and Scholtes (2000) have shown that strong stationarity implies B-stationarity. However, the reverse is true only, if the MPEC satisfies an MPEC linear independence constraint qualification (see remark following Definition 4.1) or if $\mathcal{D}^* = \emptyset$. If $\mathcal{D}^* = \emptyset$, then all stationarity concepts are equivalent. However, in the interesting case where $\mathcal{D}^* \neq \emptyset$, it follows that A-, C- and M-stationary points allow trivial descend directions, making these stationarity concepts too weak to be useful.

The first example, due to Scheel and Scholtes (2000), illustrates the failure of C-stationarity to exclude descend directions. Consider

$$\underset{x,y}{\text{minimize}} \ (x-1)^2 + (y-1)^2 \quad \text{subject to} \ \ 0 \leq x \ \perp \ y \geq 0,$$

and observe that $(0,0)$ is a C-stationary point. Since the multipliers $\nu = \mu = -2 < 0$, however, the objective can clearly be reduced by increasing either x or y; see Figure 2 (left).

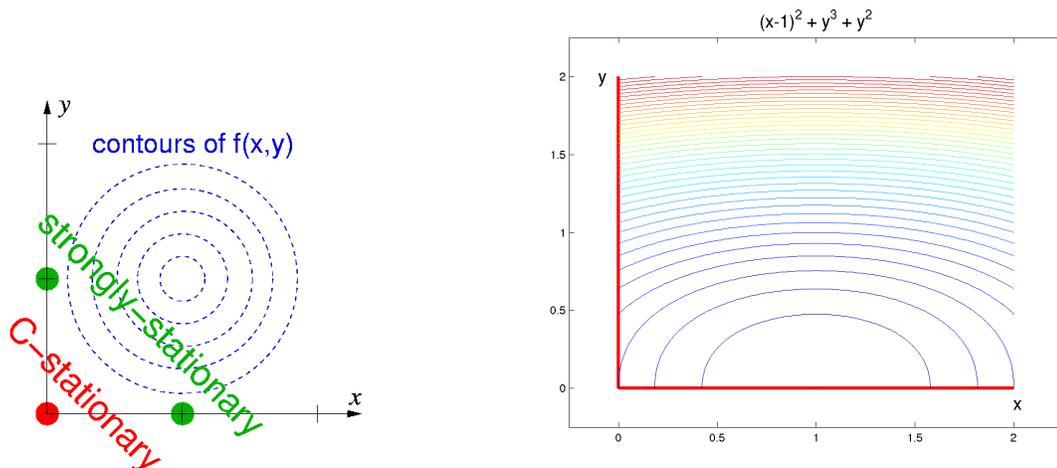Figure 2: Example with a C-stationary point with two descend directions (left), and example with an A- and M-stationary point with two descend directions(right).

Similarly, one can construct trivial examples showing that A- and M-stationary points do not preclude trivial descend directions. Consider

$$\underset{x,y}{\text{minimize}} \ (x-1)^2 + y^3 + y^2 \quad \text{subject to} \ \ 0 \leq x \ \perp \ y \geq 0. \tag{2.7}$$

In this case, the origin is A- and M-stationary, but again there exists a trivial descend direction, namely, $(1, 0)$, that reduces the objective by increasing x; see Figure 2 (right).

Remark 2.1. *We refer to so-called A-, C-, and M-stationary points as* spurious stationary points, *because they do not exclude the existence of first-order descend directions.*

In the remainder of this section, we show that many popular methods for solving MPECs are attracted to such spurious stationary points, motivating the development of new methods that avoid this pitfall.

2.2 Failures of the NLP Approaches

The NLP approach to MPECs reformulates the complementarity constraint as a set of nonlinear inequalities:

$$0 \leq y \perp F(x, y) \geq 0 \Leftrightarrow \begin{cases} y \geq 0 \\ s - F(x, y) = 0 \\ s \geq 0 \\ y^T s \leq 0. \end{cases} \tag{2.8}$$

There exists a range of other formulations that replace $y^T s \leq 0$ by a system of inequalities or various NCP functions, see, for example, (Leyffer, 2006). Our observations generalize to these other NLP approaches. The introduction of the slack variables s is necessary to avoid convergence to nonstationary points; see (Fletcher et al., 2006, Example 7.2). Similarly, the complementarity condition $y^T s \leq 0$ should not be written as an equation $y^T s = 0$, which would degrade the speed of convergence of SQP methods to strongly stationary points.

Next, we consider how NLP solvers behave when applied to the examples of the previous subsection. We start by considering an SQP method applied to the first example and started at $(x, y) = (1, 1)$, the unconstrained minimum. We observe that SQP generates the sequence $(x^{(k)}, y^{(k)}) = (\frac{1}{2^k}, \frac{1}{2^k})$ that converges linearly to the spurious stationary point $(0, 0)$, while the multipliers diverge to infinity.

The second example shows that SQP methods may converge to so-called A- or M-stationary points. Proposition 2.1 below shows that starting at $(x_0, y_0) = (0, t)$ for any $t > 0$, SQP generates a sequence of iterates that converges quadratically to the spurious stationary point $(0, 0)$.

In general, replacing the structural complementarity constraint by a set of equations prevents the solvers from "seeing around corners." Thus, without modification, the NLP approaches are doomed to converge to spurious stationary points. One could imagine remedies that monitor the sign of the complementarity multipliers, but they would require sophisticated active-set strategies that may interfere with the performance of the NLP solvers and are not readily implemented.

2.3 A Counterexample for SQPEC

Recently, Scholtes (2004) has considered optimization problems with combinatorial structure. One such example involves MPECs, where the complementarity constraint is the combinatorial structure. Scholtes suggests an SQP-like method that respects the combinatorial structure and shows local quadratic convergence under reasonable assumptions. In the MPEC case, the method is a sequential quadratic programming with equality constraints (SQPEC) method. Can this method avoid convergence to spurious stationary points? Unfortunately, the answer to this question is no, as the following proposition shows.

Proposition 2.1. *Consider solving the MPEC (2.7) by applying SQPEC (Scholtes, 2004). Starting at* $(x_0, y_0) = (0, t)$ *for* $0 < t < 1$, *SQPEC generates the following sequence of iterates,*

$$(x^{(k+1)}, y^{(k+1)}) = \left(0, \frac{3y^{(k)^2}}{6y^{(k)} + 2} \right),$$

which converges quadratically to the spurious M-stationary point $(0, 0)$.

Proof. The gradient and Hessian of (2.7) are given by

$$\nabla f = \begin{pmatrix} 2(x - 1) \\ 3y^2 + 2y \end{pmatrix} \text{ and } \nabla^2 \mathcal{L} = \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 6y + 2 \end{bmatrix},$$

respectively. Thus, the QPEC at $(x^{(k)}, y^{(k)})$ is given by

$$\begin{aligned} \underset{d}{\text{minimize}} \quad & 2(x^{(k)} - 1)d_x + (3y^{(k)^2} + 2y^{(k)})d_y + d_x^2 + (3y^{(k)} + 2)d_y^2 \\ \text{subject to} \quad & 0 \le d_x \perp y^{(k)} + d_y \ge 0. \end{aligned}$$

One can show that $d_x = 0$ is a solution of this QPEC. In this case, it follows that

$$d_y = -\frac{3y^{(k)^2} + 2y^{(k)}}{6y^{(k)} + 2} \Rightarrow y^{(k+1)} = y^{(k)} + d_y = \frac{3y^{(k)^2}}{6y^{(k)} + 2} > 0.$$

The last inequality shows that this is a local solution of the QPEC. Convergence to $(0, 0)$ follows by taking the limits, and convergence is clearly quadratic.

One can see that $(0, 0)$ is a spurious stationary point, by observing that $\nabla f(0, 0) = (-2, 0)$, which clearly indicates the existence of a descend direction if x is increased from zero. The unique B-stationary point is $(x^*, y^*) = (1, 0)$. □

SQPEC converges to the spurious stationary limit $(0, 0)$; but because it never gets there in finite time, it cannot "look around the corner" to discover the descend direction $(1, 0)$ that would allow convergence to the B-stationary solution $(1, 0)$. We note that this is *not* a counterexample to the results by Scholtes (2004), because Scholtes investigates only local quadratic convergence. However, we still believe that this example highlights a potential deficiency in an SQPEC method. In contrast, our SLPEC-EQP method would take one step to the origin, solve an EQP, and then escape from the origin at the next iteration and converge after one further EQP to the solution.

To our knowledge, currently no practical method guarantees convergence to B-stationary points under reasonable assumptions. The only exception is the branch-and-bound method proposed by Bard (1988), which is impractical even for medium-sized problems, because of the lack of suitable cutting planes. The aim of this paper is to present a new method that fills this gap.

3   Algorithm Statement

In this section we define the key components of the algorithm and provide a formal algorithm statement. We start by introducing the SLPEC method; later, we will indicate how EQP steps can be included. This simplification is consistent with our global convergence result, which relies entirely on LPEC steps.

3.1 Outline of SLPEC-EQP Algorithm

We start by defining the subproblems solved by our method and provide a rough outline of the SLPEC-EQP method. At each iteration, we solve an LPEC inside a trust region of radius $\rho > 0$ around the current point $z = (x, y)$:

$$\text{LPEC}(z, \rho) \begin{cases} \underset{d}{\text{minimize}} & g(z)^T d \\ \text{subject to} & c(z) + A(z)^T d \geq 0, \\ & 0 \leq y + d_y \perp F(z) + B(z)^T d \geq 0, \\ & \|d\| \leq \rho \end{cases}$$

where $g(z) = \nabla f(z)$, $A(z) = \nabla c(z)$, and $B(z) = \nabla F(z)$. Given a solution $d \neq 0$, we find the active sets that are predicted by the LPEC:

$$\mathcal{A}_c(z + d) \quad := \quad \{i : c_i(z) + a_i(z)^T d = 0\} \tag{3.1}$$

$$\mathcal{A}_y(z + d) \quad := \quad \{j : y_j + d_j = 0\} \tag{3.2}$$

$$\mathcal{A}_F(z + d) \quad := \quad \{j : F_j(z) + b_j(z)^T d = 0\} \tag{3.3}$$

and solve the corresponding EQP:

$$\text{EQP}(z + d) \begin{cases} \underset{d}{\text{minimize}} & g(z)^T d + \frac{1}{2} d^T H(z) d \\ \text{subject to} & c_i(z) + a_i(z)^T d = 0, \quad \forall i \in \mathcal{A}_c(z + d) \\ & y_j + d_j = 0, \quad \forall j \in \mathcal{A}_y(z + d) \\ & F_j(z) + b_j(z)^T d = 0, \quad \forall j \in \mathcal{A}_F(z + d). \end{cases}$$

We note that $\text{EQP}(z + d)$ can be solved as a linear system of equations. Global convergence is promoted through the use of a three-dimensional filter that separates the complementarity error and the nonlinear infeasibility. A conceptual outline of our proposed algorithm is given below.

Outline of SLPEC-EQP Algorithm

Given an initial point $z_0 = (x_0, y_0)$, set $k = 0$, and $\rho_0 > 0$.

while $d \neq 0$ **do**

    Solve $\text{LPEC}(z^{(k)}, \rho_k)$ for step $d^{(k)}$

    Identify the active sets $\mathcal{A}_c(z^{(k)} + d^{(k)})$, $\mathcal{A}_y(z^{(k)} + d^{(k)})$, and $\mathcal{A}_F(z^{(k)} + d^{(k)})$.

    Solve $\text{EQP}(z + d)$ for second-order step d_{qp}.

    if $z^{(k)} + d_{qp}$ *acceptable step* **then**

        Set $z^{(k+1)} := z^{(k)} + d_{qp}$, and possibly increase $\rho_{k+1} = 2\rho_k$.

    else

        Set $z^{(k+1)} := z^{(k)}$, and decrease $\rho_{k+1} = \rho_k / 2$.

    end

end

The algorithm outlined above leaves a number of important open questions: How should the LPEC be solved? What constitutes acceptance of a step? Most importantly, what happens if the LPEC or the EQP has no solution? In a practical implementation we might also restrict the EQP step by a trust region or

a proximal-point term, and we could use the SLPEC step if the EQP step fails, or we could consider a piecewise line-search along an arc.

Our SLPEC-EQP method has one important advantage over the recent NLP approaches. The solution of the LPEC matches exactly the definition of B-stationarity (see Section 2), and we therefore always work with the correct tangent cone. In particular, if $d = 0$ solves the LPEC for some $\rho > 0$, then we can conclude that the current point is B-stationary. To our knowledge, this is the only algorithm that guarantees global convergence to B-stationary points.

We will start by analyzing the global convergence properties of an SLPEC method. The SLPEC-EQP method will inherit the global convergence properties if we ensure that the EQP step realizes at least a fraction of the progress predicted by the LPEC step. A detailed description of the SLPEC-EQP method is given in Section 5.

3.2 Definitions and Notation

Our SLPEC method uses a filter (Fletcher and Leyffer, 2002; Fletcher et al., 2002a) to promote global convergence to B-stationary points. Filter methods promote convergence by viewing an optimization problem as a bi-objective optimization problem in which both the objective and the constraint violation are minimized. Unlike traditional filter methods, however, our SLPEC filter has three components. For a point $z^{(l)} := (x^{(l)}, y^{(l)})$ we define

$$
\begin{aligned}
f^{(l)} &:= f(z^{(l)}), &\text{(3.4a)} \\
h_c^{(l)} &:= h_c(z^{(l)}) := \|c(z^{(l)})^+\|_\infty, &\text{(3.4b)} \\
h_F^{(l)} &:= h_F(z^{(l)}) := \|\min(y^{(l)}, F(z^{(l)}))\|_\infty &\text{(3.4c)}
\end{aligned}
$$

to measure the objective value, the infeasibility of the general constraints, and the complementarity constraint violation respectively. In the definition of $h_c^{(l)}$, we have used the notation $a^+ := \max(0, a)$. The use of two infeasibility measures gives us greater flexibility to define a restoration phase later. For convenience, we also define the total constraint violation as

$$
h(z) := \max\left(h_c(z), h_F(z)\right) := \max\left(\|c(z)^+\|_\infty \, , \, \|\min(y, F(z))\|_\infty\right). \tag{3.5}
$$

We note that we could have chosen any other norm to measure the constraint violation. The ℓ_∞-norm has the advantage of simplifying the constants in the convergence proofs.

Definition 3.1. *A filter is defined as follows:*

1. *A point $z^{(k)} := (x^{(k)}, y^{(k)})$ is said to* dominate *another point $z^{(l)} := (x^{(l)}, y^{(l)})$ if the triple $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ dominates $(h_c^{(l)}, h_F^{(l)}, f^{(l)})$, namely,*

$$
f^{(k)} < f^{(l)}, \ h_c^{(k)} < h_c^{(l)}, \ \text{and} \ h_F^{(k)} < h_F^{(l)}.
$$

2. *A filter for (1.1) is a list $\mathcal{F}$ of triples $(h_c^{(l)}, h_F^{(l)}, f^{(l)})$ such that no triple dominates any other triple for all $l \in \mathcal{F}$.*

3. *We also define minimum the total constraint violation for a filter:*

$$
\tau := \min_{l \in \mathcal{F}} \left\{ h^{(l)} \right\}.
$$

Dominance alone is not sufficient to ensure convergence of nonlinear solvers. In practice, we need to add a small margin around the filter to ensure convergence.

Definition 3.2. *We say that a point (x, y) is acceptable to the filter $\mathcal{F}$ if its corresponding triple (h_c, h_F, f) satisfies*

$$f \leq f^{(l)} - \gamma h^{(l)}, \;\; or \;\; h_c \leq h_c^{(l)} - \gamma h^{(l)}, \;\; or \;\; h_F \leq h_F^{(l)} - \gamma h^{(l)}, \; \forall l \in \mathcal{F}, \tag{3.6}$$

where $0 < \gamma < 1$ is a small constant.

We note, that this three-dimensional definition differs from the usual two-dimensional filter in the sense that the margin depends on the total constraint violation, rather than on the individual constraint violation. This change allows us to prove convergence to feasible limit points in the next section. If we relax the final two conditions in (3.6) to $h_c \leq (1 - \gamma)h_c^{(l)}$ or $h_F \leq (1 - \gamma)h_F^{(l)}$, then we can no longer show that the limit point is feasible, as it may be feasible with respect to one of the infeasibility measures, but not both.
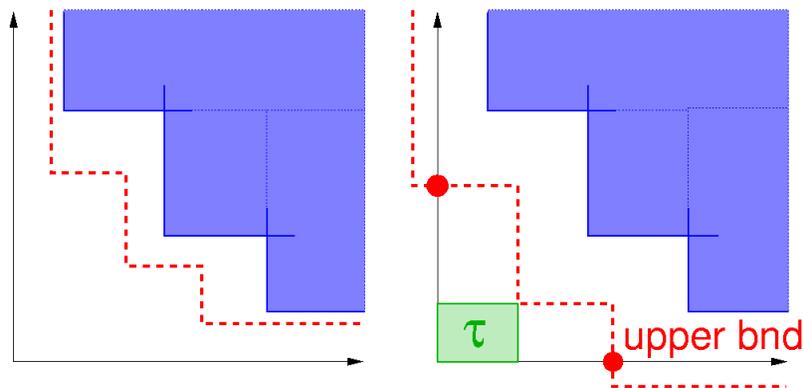

Figure 3: Usual filter envelope (left) and filter envelope used in (3.6) (right).

Figure 3 shows the difference between the traditional filter envelope and the new filter envelope, which has also been used in (Gould et al., 2004). The dashed line shows the margins. By changing the margin to be proportional to the total constraint violation, we introduce implicit bounds on the constraint violation in $h_c(z)$ and $h_F(z)$.

The filter ensures convergence only to a feasible limit point, and we require a sufficient reduction condition to ensure convergence to stationary points. We define the predicted and actual reduction in the objective function over a step d as

$$\Delta f := f(z) - f(z + d) \;\; \text{and} \;\; \Delta l := -\nabla f(z)^T d,$$

respectively. However, we cannot expect that a sufficient reduction condition holds for points that are far from the feasible set. This observation motivates the introduction of a so-called switching condition (Wächter and Biegler, 2005b,a), which switches on a sufficient reduction condition, whenever we are close to a feasible point. Formally, we require that

$$\Delta f \;\; \geq \;\; \sigma \Delta l \;\; \text{whenever} \tag{3.7}$$
$$\Delta l \;\; \geq \;\; \delta h^2 \tag{3.8}$$

holds, where $\delta > 0$ and $0 < \gamma \leq \sigma < 1$ are constants.

A successful iteration of SLPEC for which (3.7) and (3.8) holds is called an *f-type* iteration, and a successful iteration for which (3.8) does not hold is called an *h-type* iteration to indicate that the primary purpose of the step is to decrease the constraint violation.

Our algorithm also requires a restoration phase if LPEC(z, ρ) is inconsistent, which may happen far from the feasible set or because the trust-region parameter becomes too small. One can define a restoration problem in several ways. For example,

$$
\begin{aligned}
\underset{x,y}{\text{minimize}} \quad & \|c(x, y)^+\| \\
\text{subject to} \quad & 0 \leq y \perp F(x, y) \geq 0
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\underset{x,y,s}{\text{minimize}} \quad & \|c(x, y)^+\| + \|s - F(x, y)\| \\
\text{subject to} \quad & 0 \leq y \perp s \geq 0
\end{aligned}
\tag{3.10}
$$

are two possibilities. Problem (3.9) aims to reduce the general constraint violation $h_c(z)$ and may be more suitable if we can guarantee the existence of solutions to the complementarity constraint for all x. Problem (3.10) is more suitable whenever we cannot ensure that the complementarity constraint can be satisfied. Both problems can be formulated as smooth LPECs, and the general algorithm proposed below can be used to solve these problems. SLPEC will maintain feasibility of linear complementarity if started at a feasible point, which implies that (3.10) does not require a recursive restoration phase.

3.3   An SLPEC-Filter Algorithm for MPECs

We formally state the SLPEC algorithm in pseudo-code below. The algorithm has an inner and an outer loop. The inner loop reduces the trust-region radius until either an acceptable point is found or the problem becomes inconsistent, in which case we enter the restoration phase. The outer loop generates the sequence of iterates $z^{(k)} = (x^{(k)}, y^{(k)})$.

4   SLPEC-Filter Convergence Proof

This section establishes convergence to B-stationary point of our SLPEC-Filter algorithm. The extension to an SLPEC-EQP procedure is described in Section 5.

4.1   Preliminaries

We start with some preliminary results. The disjunctive nature of MPECs means that every LPEC consists of a finite collection of LP-pieces. To derive a more suitable version of B-stationary for our analysis, we define the following active sets:

$$
\begin{aligned}
\mathcal{A}_c(z) \quad &:= \quad \{i \,|\, c_i(z) = 0\} & (4.1) \\
\mathcal{A}_y(z) \quad &:= \quad \{j \,|\, y_j = 0\} & (4.2) \\
\mathcal{A}_F(z) \quad &:= \quad \{j \,|\, F_j(z) = 0\} \,. & (4.3)
\end{aligned}
$$

The set of degenerate complementarity constraints is now given by

$$
\mathcal{D}(z) := \mathcal{A}_y(z) \cap \mathcal{A}_F(z).
$$

SLPEC Algorithm

Given (x_0, y_0), $\rho \in [\underline{\rho}, \overline{\rho}]$, set $k = 0$; compute $\nabla f^{(k)}, \nabla c^{(k)}, \nabla F^{(k)}$

while *not optimal* **do**

    reset trust-region radius $\rho \in [\underline{\rho}, \overline{\rho}]$

    repeat

        solve LPEC$(z^{(k)}, \rho)$ for a first-order step d

        if $\exists$ *solution d of LPEC$(z^{(k)}, \mu)$* **then**

            if $d = 0$ **then terminate** B-stationary point found

            compute predicted reduction Δl

            evaluate $f(z^{(k)} + d)$, $h_c(z^{(k)} + d)$, and $h_F(z^{(k)} + d)$

            if *$z^{(k)} + d$ acceptable to filter and $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$* **then**

                if $\Delta l^{(k)} < \delta(h^{(k)})^2$ **then**

                    set $\rho_k = \rho$, $d^{(k)} = d$, $\Delta l^{(k)} = \Delta l$, $\Delta f^{(k)} = \Delta f$

                    add $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ to the filter   *h-type iteration*

                else if $\Delta f \geq \sigma \Delta l$ *and* $\Delta l \geq \delta(h^{(k)})^2$ **then**

                    set $\rho_k = \rho$, $d^{(k)} = d$, $\Delta l^{(k)} = \Delta q$, $\Delta f^{(k)} = \Delta f$ *f-type iteration*

                else

                  reduce trust-region radius $\rho = \rho/2$

            else

               reduce trust-region radius $\rho = \rho/2$

        else

            add $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ to filter

            enter **restoration phase** to find acceptable/compatible point, $z^{(k+1)}$

    until *new $z^{(k+1)}$ found*

    set $k = k + 1$, update gradients $\nabla f^{(k)}, \nabla c^{(k)}, \nabla F^{(k)}$ & test for convergence

---

In addition, we define the set of binding complementarity constraints, namely, those where strict complementarity holds and either $y_j = 0$ or $F_j(z) = 0$ (but not both):

$$\mathcal{A}_{y+}(z) \;\; := \;\; \{j \in \mathcal{A}_y(z) | F_j(z) > 0\} \tag{4.4}$$

$$\mathcal{A}_{F+}(z) \;\; := \;\; \{j \in \mathcal{A}_F(z) | y_j > 0\}. \tag{4.5}$$

Now we can state an equivalent condition for B-stationarity.

Proposition 4.1. *A point z^* is B-stationary if and only if $d = 0$ solves the collection of LPs given by*

$$
\begin{aligned}
\underset{d}{\text{minimize}} \quad & {g^*}^T d \\
\text{subject to} \quad & c^* + {A^*}^T d \geq 0, \\
& F_i^* + {b_i^*}^T d \geq 0 \quad \text{and} \quad d_i = 0, \qquad \forall i \in \mathcal{A}_{y+}^* \\
& F_i^* + {b_i^*}^T d = 0 \quad \text{and} \quad y_i^* + d_i \geq 0, \quad \forall i \in \mathcal{A}_{F+}^* \\
& F_i^* + {b_i^*}^T d \geq 0 \quad \text{and} \quad d_i = 0, \qquad \forall i \in \mathcal{D}_y \\
& F_i^* + {b_i^*}^T d = 0 \quad \text{and} \quad y_i^* + d_i \geq 0, \quad \forall i \in \mathcal{D}_F,
\end{aligned}
$$

for every partition $(\mathcal{D}_y, \mathcal{D}_F)$ of $\mathcal{D}^*$, that is, $\mathcal{D}_y \cap \mathcal{D}_F = \emptyset$ and $\mathcal{D}_y \cup \mathcal{D}_F = \mathcal{D}^*$.

The proof follows readily by observing that the partitions of $\mathcal{D}^*$ represent the disjunctive nature of the complementarity condition. Proposition 4.1 indicates a potential computational inefficiency of our method: at every step, we may have to solve 2^d LPs, where $d = |\mathcal{D}^{(k)}|$ is the number of different LP pieces. However, this assessment turns out to be overly pessimistic, as we indicate in Section 5.

Our convergence result uses a piecewise Mangasarian-Fromowitz constraint qualification (MFCQ). We include its definition for the sake of completeness; see also (Scholtes, 2001).

Definition 4.1. *We say that the MPEC (1.1) satisfies an MPEC-MFCQ if and only if, for every partition $(\mathcal{D}_y, \mathcal{D}_F)$ of $\mathcal{D}^*$, the standard NLP defined as*

$$
\begin{aligned}
\underset{(x,y)}{\text{minimize}} \quad & f(x,y) \\
\text{subject to} \quad & c(x,y) \geq 0, \\
& y_i = 0 \ \text{ and } \ F_i(x,y) \geq 0, \quad \forall i \in \mathcal{A}^*_{y+} \\
& y_i \geq 0 \ \text{ and } \ F_i(x,y) = 0, \quad \forall i \in \mathcal{A}^*_{F+} \\
& y_i = 0 \ \text{ and } \ F_i(x,y) \geq 0, \quad \forall i \in \mathcal{D}_y \\
& y_i \geq 0 \ \text{ and } \ F_i(x,y) = 0, \quad \forall i \in \mathcal{D}_F
\end{aligned}
$$

satisfies an MFCQ.

We note that a linear-independence constraint qualification (LICQ) for MPECs can be defined in a similar way. Next, we generalize Fritz-John necessary optimality conditions to MPECs, by applying Fritz-John conditions to every LP-piece.

Proposition 4.2. *Let z^* solve the MPEC (1.1), and assume that (1.1) satisfies an MPEC-MFCQ. Then the following hold:*

1. *z^* is a feasible point, that is $c(z^*) \geq 0$, and $0 \leq y^* \perp F(z^*) \geq 0$.*

2. *For every partition $(\mathcal{D}_y, \mathcal{D}_F)$ of $\mathcal{D}^*$, the following holds:*

$$
\begin{aligned}
& \big\{ s \,|\, {g^*}^T s < 0 && &&\text{(4.6a)} \\
& \qquad {a_i^*}^T s > 0 && \forall i \in \mathcal{A}^*_c && \text{(4.6b)} \\
& \qquad\quad s_i = 0 && \forall i \in \mathcal{A}^*_{y+} && \text{(4.6c)} \\
& \qquad {b_i^*}^T s = 0 && \forall i \in \mathcal{A}^*_{F+} && \text{(4.6d)} \\
& s_i = 0 \ \text{ and } \ {b_i^*}^T s > 0 && \forall i \in \mathcal{D}_y && \text{(4.6e)} \\
& s_i > 0 \ \text{ and } \ {b_i^*}^T s = 0 && \forall i \in \mathcal{D}_F \,\big\} = \emptyset. && \text{(4.6f)}
\end{aligned}
$$

Proof. The proof follows from the Fritz-John conditions on every LP piece, assuming that an MPEC-MFCQ holds. □

A consequence of Proposition 4.2 is that if z^∞ is a feasible point that is *not optimal*, then there exists a

partition $(\mathcal{D}_y, \mathcal{D}_F)$ of $\mathcal{D}^\infty$ such that the set

$$\left\{ s \,|\, {g^\infty}^T s < 0 \right. \tag{4.7a}$$

$${a_i^\infty}^T s > 0 \qquad \forall i \in \mathcal{A}_c^\infty \tag{4.7b}$$

$$s_i = 0 \qquad \forall i \in \mathcal{A}_{y+}^\infty \tag{4.7c}$$

$${b_i^\infty}^T s = 0 \qquad \forall i \in \mathcal{A}_{F+}^\infty \tag{4.7d}$$

$$s_i = 0 \text{ and } {b_i^\infty}^T s > 0 \qquad \forall i \in \mathcal{D}_y \tag{4.7e}$$

$$\left. s_i > 0 \text{ and } {b_i^\infty}^T s = 0 \qquad \forall i \in \mathcal{D}_F \right\} \neq \emptyset \tag{4.7f}$$

is not empty. In other words there exists an LP-piece that has a strictly interior descend direction along which the objective can be reduced. Thus, there exist $\epsilon > 0$ and a direction s with $\|s\| = 1$ and a partition $(\mathcal{D}_y, \mathcal{D}_F)$ of $\mathcal{D}^\infty$ such that

$$ {g^\infty}^T s \leq -\epsilon \tag{4.8a}$$

$${a_i^\infty}^T s \geq \epsilon \qquad \forall i \in \mathcal{A}_c^\infty \tag{4.8b}$$

$$s_i = 0 \qquad \forall i \in \mathcal{A}_{y+}^\infty \tag{4.8c}$$

$${b_i^\infty}^T s = 0 \qquad \forall i \in \mathcal{A}_{F+}^\infty \tag{4.8d}$$

$$s_i = 0 \text{ and } {b_i^\infty}^T s \geq \epsilon \qquad \forall i \in \mathcal{D}_y \tag{4.8e}$$

$$s_i \geq \epsilon \text{ and } {b_i^\infty}^T s = 0 \qquad \forall i \in \mathcal{D}_F. \tag{4.8f}$$

We will exploit this existence of descend directions in our convergence analysis.

4.2 Assumptions

To derive our convergence results, we make the following assumptions on the MPEC problem.

Assumption 4.1. *The iterates remain in a compact set, Z.*

Assumption 4.2. *The problem functions f, c, and F are twice continuously differentiable on an open set containing Z.*

Assumption 4.3. *Any limit point satisfies an MPEC-MFCQ, see Definition 4.1.*

These assumptions are quite mild. In particular, we do not assume that an MPEC-LICQ holds, an assumption that is unreasonable in practice. The strongest assumption is Assumption 4.1, because it may be difficult to derive bounds on the variables y. However, there exist sufficient conditions that ensure that such bounds exist. For example, if the MPEC arises out of a bilevel optimization problem, where the lower-level problem satisfies an MFCQ, then the multipliers of the lower-level problem are bounded. Thus, as long as the primal variables are bounded, the dual variables are also bounded, and the variables of the MPEC are bounded.

In addition, we require an assumption on the quality of the solution of each LPEC that is similar in spirit to a Cauchy-point condition. This assumption is not needed in the case of standard SLP-EQP filter methods because we can usually solve LPs to global optimality. We require this additional assumption for MPECs, however, to handle the disjunctive nature of each LPEC. We are concerned with the situation where our

algorithm approaches a spurious stationary point but when solving the LPEC, always chooses the wrong partition of $\mathcal{D}^{(k)}$ for which the predicted reduction approaches zero. In that case, we may get stuck at a spurious stationary point. To avoid this situation, we make the following assumption on the solution of the LPEC.

Assumption 4.4. *The method for solving the LPEC approximation explores all ϵ-active complementarity partitions of $\mathcal{D}^{(k)}$ and chooses the largest predicted reduction from among these.*

This assumption does not mean that the LPEC is solved to global optimality (though this would be a sufficient condition). It requires only the solution of all LP pieces at the starting point. In Section 6 we give a computationally more efficient way to guarantee global convergence. We note that a sufficient condition for Assumption 4.4 is that each LPEC is solved to global optimality.

4.3   Main Convergence Result

Our main convergence result shows that the SLPEC method generates a subsequence that converges to a stationary point or a local minimum of the infeasibility measure $h(z)$.

Theorem 4.1. *Let Assumptions 4.1–4.4 hold. Then it follows that the SLPEC-filter algorithm terminates with one of the following mutually exclusive outcomes.*

O1 *The algorithm terminates at a B-stationary point; that is $d = 0$ solves LPEC($z^{(k)}, \rho$) for some k.*

O2 *The algorithm generates an infinite sequence of iterates that has an accumulation point that is feasible and B-stationary.*

O3 *The restoration phase fails to find a point that is acceptable to the filter.*

Outcomes **O1** and **O2** correspond to normal termination of the algorithm. If the limit point fails to satisfy an MPEC-MFCQ, then we can no longer guarantee B-stationarity, but the limit remains feasible. Outcome **O3** corresponds to the case where the complementarity constraints and/or the general constraints are locally inconsistent. Unless we make very restrictive assumptions on the class of problem that we consider, this outcome cannot be excluded.

Outline of Convergence Proof. We start by showing that feasibility of the LPEC implies bounds on the predicted reduction and the infeasibility after the LPEC step (Lemma 4.1). Next, we show that in a neighborhood of a feasible but not stationary point, the LPEC will generate a step that is acceptable to the filter and reduces the objective function, resulting in an f-type step (Lemma 4.2). This lemma allows us to show that the inner iteration terminates finitely (Lemma 4.3). Thus, the algorithm generates an infinite sequence, and we show that there exists a limit point that is feasible (Lemma 4.4). Finally, we prove Theorem 4.1 by considering the two mutually exclusive cases: an infinite number or a finite number of h-type steps.

4.4   SLPEC-Filter Convergence Proof

Our convergence proof follows the filter convergence proofs of Chin and Fletcher (2003) and Fletcher et al. (2002b), though it requires extra care to handle the disjunctive nature of the subproblems and the fact that we are using a three-dimensional filter. We start by extending a lemma about properties of the LPEC step.

Lemma 4.1. *Let $M > 0$ be a constant such that*

$$\|s^T \nabla f(z)s\| \leq M, \ \|s^T \nabla F_j(z)s\| \leq M, \ \text{and} \ \|s^T \nabla c_i(z)s\| \leq M, \quad \forall s : \|s\|_\infty = 1$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, q$, and let $d \neq 0$ solve LPEC($z^{(k)}, \rho$). Then it follows that for all $i = 1, \ldots, m$ and $j = 1, \ldots, q$

$$c_i(z^{(k)} + d) \ \geq \ -\rho^2 M \tag{4.9a}$$
$$h_c(z^{(k)} + d) \ \leq \ \rho^2 M \tag{4.9b}$$
$$|F_j(z^{(k)} + d)| \ \leq \ \rho^2 M \tag{4.9c}$$
$$y^{(k)} + d_y \ \geq \ 0 \tag{4.9d}$$
$$h_F(z^{(k)} + d) \ \leq \ \min(\rho|Z|, \rho^2 M) \tag{4.9e}$$
$$\Delta f \ \geq \ \Delta l - \rho^2 M, \tag{4.9f}$$

where $|Z|$ is the radius of the bounded set Z.

Proof. Taylor's theorem implies that there exists a point ξ_i along the line segment from $z^{(k)}$ to $z^{(k)} + d$ such that

$$c_i(z^{(k)} + d) = c_i^{(k)} + a_i^{(k)^T} d + \frac{1}{2} d^T \nabla^2 c_i(\xi_i) d \geq -\rho^2 M,$$

where the inequality follows from the feasibility of LPEC($z^{(k)}, \rho$) and the fact that $\|d\|_\infty \leq \rho$. The bound (4.9b) follows from the definition of the constraint violation (3.4).

The bound (4.9c) can be shown similar to the bound (4.9a), and (4.9d) follows from the feasibility of the LPEC. To prove (4.9d), we distinguish three cases. If $y_j^{(k)} + d_j = 0$, then we conclude that the minimum of component j is bounded by $\rho^2 M$ from (4.9c). If $y_j^{(k)} + d_j > 0$ and $F_j(z^{(k)} + d) < 0$, then the bound follows again from (4.9c); and if $y_j^{(k)} + d_j > 0$ and $F_j(z^{(k)} + d) > 0$, then the minimum of component j is bounded by $\min(|Z|\rho, \rho^2 M)$.

Taylor's theorem implies that there exists ξ along the line segment from $z^{(k)}$ to $z^{(k)} + d$ such that

$$f(z^{(k)} + d) = f^{(k)} + g^{(k)^T} d + \frac{1}{2} d^T \nabla^2 f(\xi) d.$$

Rearranging this equation and exploiting the definition $\Delta l = -g^{(k)^T} d$, we have that

$$\Delta f = \Delta l - \frac{1}{2} d^T \nabla^2 f(\xi) d \geq \Delta l - \rho^2 M.$$

$\square$

Next, we show that near a feasible and nonstationary point, the LPEC step will be a filter-acceptable f-type step.

Lemma 4.2. *Let z^∞ be a feasible but not stationary point. Then there exist a neighborhood $\mathcal{N}^\infty$ of z^∞ and constants $\epsilon > 0$, $\kappa > 0$, and $\mu > 0$ such that for any $z \in \mathcal{N}^\infty$ the LPEC(z, ρ) is compatible and produces a filter-acceptable f-type step for all trust-region radii ρ in the range*

$$\frac{h(z)}{\epsilon} \ \leq \ \rho \ \leq \ \kappa. \tag{4.10}$$

Proof. Let $z \in \mathcal{N}^\infty$. We start by showing that the LPEC(z, ρ) is compatible. Consider the equality constraints in LPEC(z, ρ) induced by the complementarity constraint. Because the MPEC satisfies an MPEC-MFCQ, it follows that the constraint normals $e_j, b_k(z)$ for $j \in \mathcal{A}_F^\infty$ and $k \in \mathcal{A}_y^\infty$ are linearly independent. We denote the basis matrix by $B := [e_j : b_k(z)]$, and its generalized inverse by $B^+ := (B^T B)^{-1} B^T$. The closest point to the linearized equality constraints to $d = 0$ is given by $p = -B^+ l(z)$, where $l(z)$ is the right-hand side of the linearized equality constraints $y_j + d_j = 0$ for $j \in \mathcal{A}_F^\infty$ and $F_k(z) + b_k(z)^T d = 0$ for $k \in \mathcal{A}_y^\infty$. We denote the length of p by $\hat{p} := \|p\|_2$ and observe that $\hat{p} = \mathcal{O}(h(z))$ from the definition of p. We can therefore choose $\mu > 0$ such that

$$\rho > \hat{p} = \mathcal{O}(h(z)) = \mu h(z). \tag{4.11}$$

Because z^∞ is not stationary, there exists $\epsilon > 0$ and a direction s^∞ with $\|s^\infty\|_2 = 1$ such that (4.8) holds. We now form the closest unit vector to s^∞ in the null space of B^T as

$$s = (I - BB^+) s^\infty \,/\, \|(I - BB^+) s^\infty\|_2$$

and observe that there exists a possibly smaller neighborhood $\mathcal{N}^\infty$ such that

$$
\begin{align}
s^T g &\leq -\epsilon \tag{4.12}\\
s^T a_i &\geq \epsilon \quad \forall i \in \mathcal{A}_c^\infty \tag{4.13}\\
s^T b_i &\geq \epsilon \quad \forall i \in \mathcal{A}_F^\infty \tag{4.14}\\
s_j &\geq \epsilon \quad \forall j \in \mathcal{A}_y^\infty \tag{4.15}
\end{align}
$$

holds for any $z \in \mathcal{N}^\infty$.

Now consider the solution to LPEC(z, ρ) along the line segment

$$d_\alpha = p + \alpha(\rho - \hat{p}) s \quad \text{for } \alpha \in [0, 1]$$

for fixed $\rho > \hat{p}$. It follows that d_α satisfies the equality constraints by construction. The orthogonality of p and s and $\rho > \hat{p}$ ensures that for $\alpha = 1$

$$\|d_1\| = \sqrt{\hat{p}^2 + (\rho - \hat{p})^2} = \sqrt{\rho^2 - 2\rho\hat{p} + 2\hat{p}^2} \leq \rho,$$

so that d_1 satisfies the trust-region constraint of LPEC(z, ρ).

Next, we show that we can ignore the inactive constraints, $i \notin \mathcal{A}_c^\infty$, for a suitable value of ρ. It follows that there exist constants $\bar{c}, \bar{a} > 0$ independent of ρ such that

$$c_i \geq \bar{c} \quad \text{and} \quad a_i^T s \leq \bar{a} \; \forall i \notin \mathcal{A}_c^\infty$$

for every s such that $\|s\|_2 \leq 1$ by continuity of $c_i(z)$ and boundedness of $a_i(z)$. This implies the bound

$$c_i + \rho a_i^T s \geq \bar{c} - \rho \bar{a}.$$

Thus, the inactive constraints are satisfied as long as

$$\rho \leq \frac{\bar{c}}{\bar{a}}. \tag{4.16}$$

A similar result holds for the inactive complementarity constraints, and we can adjust the constants $\bar{c}, \bar{a} > 0$ accordingly.

Now we consider the active inequality constraints (again for the sake of simplicity we consider only the active general constraints, but we note that a similar result holds for the active complementarity constraints). For $i \in \mathcal{A}_c^\infty$ we obtain from (4.8) that

$$c_i + a_i^T d_1 = c_i + a_i^T + (\rho - \hat{p})a_i^T \le c_i + a_i^T + (\rho - \hat{p})\epsilon.$$

Thus, if

$$\rho \ge \hat{p} - \frac{c_i + a_i^T p}{\epsilon}, \tag{4.17}$$

then the active constraints are also feasible. The right-hand side of this inequality is $\mathcal{O}(h(c))$. Thus, there exist $\mu > 0$ and $\kappa > 0$ such that for $\mu h(z) \le \rho \le \kappa$, LPEC$(z, \rho)$ is compatible.

Next, we consider the predicted reduction. It follows from (4.8) that

$$g^{\infty^T} d_1 = g^{\infty^T} p + (\rho - \hat{p})g^{\infty^T} s \le \mathcal{O}(\hat{p}) - (\rho - \hat{p})\epsilon.$$

Feasibility of d_1 and $\hat{p} = \mathcal{O}(h(z))$ imply that the predicted reduction satisfies $\Delta l \ge \rho\epsilon - \xi h(z)$ for some ξ sufficiently large. Thus,

$$\Delta l \ge \frac{1}{2}\rho\epsilon, \;\; \text{if} \;\; \rho \ge 2\xi h(z)/\epsilon \tag{4.18}$$

which can be achieved by making μ sufficiently large in (4.11). It follows from (4.18) and (4.9f) that

$$\frac{\Delta f}{\Delta l} \ge 1 - \frac{\rho^2 M}{\Delta l} \ge 1 - \frac{2\rho M}{\epsilon}.$$

Thus, if $\rho \le (1 - \sigma)\epsilon/(2M)$, then the sufficient reduction condition (3.7) holds. From (3.7), it follows that

$$\Delta f - \gamma h(z + d) \ge \frac{1}{2}\sigma\rho\epsilon - \gamma\rho^2 M \ge 0$$

if $\rho \le \sigma\epsilon/(2\gamma M)$.

It remains to be shown that the step is also acceptable to the filter. The mechanism of the filter ensures that $\tau > 0$, because any step starting from a point z with zero constraint violation ($h(z) = 0$) satisfies the switching condition (3.8) and is therefore an f-type step. It follows from (3.5), (4.9b), and (4.9e) that $h(z+d) \le \rho^2 M$. Thus, if $\rho \le \sqrt{\beta\tau/M}$, then $h(z+d) \le \tau$ is acceptable to the filter, where $\beta = 1 - \gamma > 0$.

Putting all the bounds on ρ together, we observe that if

$$\mu h(z) \le \rho \le \min\left(\frac{\sigma\epsilon}{2\gamma M}, \frac{(1-\sigma)\epsilon}{2M}, \frac{\bar{c}}{\bar{a}}, \sqrt{\frac{\beta\tau}{M}}\right),$$

then the LPEC(z, ρ) is consistent, and the conditions for a successful f-type step are satisfied. We note that the right-hand side of this range is a constant $\kappa > 0$, independent of ρ. □

The next lemma shows that the algorithm is well defined and that the inner iteration terminates finitely. Thus, if the algorithm does not terminate finitely, it generates an infinite sequence that has an accumulation point as a consequence of Assumption 4.1.

Lemma 4.3. *Let Assumptions 4.1–4.3 hold. Then it follows that the inner iteration terminates in a finite number of steps.*

Proof. If $z^{(k)}$ is B-stationary, then $d = 0$ solves LPEC$(z^{(k)}, \rho)$ for any $\rho > 0$, and the inner iteration terminates. Hence, in the remainder of the proof we can assume that $z^{(k)}$ is *not* B-stationary.

The proof is by contradiction. We assume that the inner iteration does not terminate finitely. Then it follows that $\rho \to 0$ from the mechanism of the algorithm. We distinguish two cases, depending on whether the current point is infeasible.

Case (1): Current point is infeasible, that is $h^{(k)} > 0$. Then there exists an index i such that $c_i^{(k)} = -h^{(k)}$, or there exists an index or j such that $|\min(y_j^{(k)}, F_j^{(k)})| = h^{(k)}$. Consider the case that $c_i^{(k)} = -h^{(k)}$. Then

$$c_i^{(k)} + a_i^{(k)T} d \le c_i^{(k)} + \rho \|a_i^{(k)}\|_1 < 0, \quad \text{for all} \ \rho \ \text{such that} \ \rho < \frac{-c_i^{(k)}}{\|a_i^{(k)}\|_1}.$$

If $\|a_i^{(k)}\|_1 = 0$, then the result holds for any $\rho > 0$. Thus, the LPEC is not consistent for ρ sufficiently small. If, on the other hand, $|\min(y_j^{(k)}, F_j^{(k)})| = h^{(k)}$, we can distinguish three cases: $F_j^{(k)} = -h^{(k)}$, $F_j^{(k)} = h^{(k)}$, and $y_j^{(k)} = h^{(k)}$. In all three cases, we again observe that the LPEC will be inconsistent for a sufficiently small ρ using a similar argument. Thus, we enter the restoration after a finite number of iterations, which contradicts the assumption that the inner iteration is infinite.

Case (2): Current point is feasible, that is $h^{(k)} = 0$. Again, if the inner iteration does not terminate finitely, then it follows that $\rho \to 0$. Because $z^{(k)}$ is not a stationary point (the algorithm would have terminated with Outcome **O1**), we can apply Lemma 4.2. Thus, the conditions for a successful f-type step are satisfied for $0 \le \rho \le \kappa$, and the inner iteration terminates finitely. □

A consequence of Lemma 4.3 is that if the algorithm does not terminate with Outcome **O1** or **O3**, then it generates an infinite sequence of iterates, and there exists an accumulation point due to Assumption 4.1. Next, we show that the filter envelope forces iterates toward a feasible point. The result is a straightforward extension of Lemma 1 in (Chin and Fletcher, 2003).

Lemma 4.4. *The SLPEC-Filter algorithm generates a feasible limit point.*

Proof. If the algorithm generates an infinite number of h-type steps, then feasibility of the subsequence on which $(h_c^{(k)}, h_F^{(k)}, f^{(K)})$ is entered into the filter follows from Lemma 1 in (Chin and Fletcher, 2003). If the algorithm generates a finite number of h-type steps, then feasibility follows from the boundedness of $f^{(k)}$ and the switching condition (3.8), which ensures that $h^{(k)} \to 0$ (otherwise, f would be unbounded below). □

We are now in a position to prove our main convergence result.

Proof of Theorem 4.1. We need to consider only Outcome **O2**, because in the other two cases we either obtain a stationary point or conclude that the constraints are locally inconsistent. The convergence proof is divided into two parts, depending on whether the algorithm generates an infinite or finite number of h-type steps.

Case 1: Algorithm 2 generates an infinite number of h-type steps. We consider the subsequence of h-type iterations and observe that $h^{(k)} \to 0$, and consequently $\tau_k \to 0$ (because only h-type steps can reset τ_k). It follows that there exists a subsequence such that $h^{(k)} - = \tau_{k+1} < \tau_k$ for which $x^{(k)} \to x^\infty$. We assume that x^∞ is not stationary and seek a contradiction. We now apply Lemma 4.2, which implies that there exists a neighborhood of x^∞ in which the conditions for an f-type step are satisfied. Thus, for k sufficiently large,

$x^{(k)} \in \mathcal{N}^\infty$, and if ρ is chosen such that

$$\mu h^{(k)} \leq \rho \leq \min\left\{\sqrt{\frac{\beta \tau_k}{M}}, \kappa\right\}, \tag{4.19}$$

then we take an f-type step at $x^{(k)}$. Observe that, in the limit, $h^{(k)} < \tau_k \to 0$ and the right-hand side in (4.19) is more than twice the left-hand side. Thus, the mechanism of the algorithm that selects $\rho \geq \bar{\rho}$ and then halves the trust region will locate a value of ρ in this range. We cannot produce an h-type step for a larger value of ρ because Δl decreases monotonically with ρ. Thus, for k sufficiently large, our iterates come from f-type steps, which contradicts the assumption that all steps are h-type steps.

Case 2: Algorithm 2 generates a finite number of h-type steps. In this case, we can assume that all iterations are f-type iterations for k sufficiently large. Thus, $(h_c^{(k+1)}, h_F^{(k+1)}, f^{(k+1)})$ is always acceptable to $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$, and the sufficient reduction condition $\Delta f^{(k)} \geq \sigma \Delta l^{(k)} > 0$ is satisfied. This implies that the sequence $\{f^{(k)}\}$ is monotonically decreasing and that $h^{(k)} \to 0$, so that the limit point z^∞ is feasible. Now assume that z^∞ is not stationary and seek a contradiction. Lemma 4.2 implies that for any ρ in the range

$$\mu h^{(k)} \leq \rho \leq \min\left\{\sqrt{\frac{\beta \tau_k}{M}}, \kappa\right\}$$

the conditions for a successful f-type step are satisfied. We note that the upper bound of this range is now a constant, say $\hat{\rho}$, because τ_k is reset only in h-type steps. Thus, the inner iteration will choose a trust-region radius $\rho \geq \min(\bar{\rho}, \hat{\rho})$, which is bounded away from zero. The sufficient reduction condition becomes

$$\Delta f^{(k)} \geq \frac{1}{2}\sigma\rho\epsilon \geq \frac{1}{2}\sigma\epsilon\min(\bar{\rho}, \hat{\rho}).$$

It follows that $f^{(k)}$ is unbounded below, which contradicts Assumptions 4.1 and 4.3. $\qquad\square$

5 Accelerating Local Convergence and Computational Considerations

In this section we present an extension of the SLPEC algorithm of the previous section that includes EQP steps, and we discuss some computational aspects of our method.

5.1 Extension to SLPEC-EQP Methods

The SLPEC algorithm defined in Section 3 and analyzed in Section 4 provides only a linear rate of convergence. However, we can easily add an EQP phase that allows us to achieve faster rate of local convergence. The resulting SLPEC-EQP algorithm is defined below in pseudo-code.

The algorithm computes a Cauchy step, which is the first minimum of the quadratic model along the LPEC-step $d = d_{lp}$ for $0 < \alpha \leq 1$:

$$q_k(\alpha d) := f^{(k)} + \alpha g^{(k)^T}d + \alpha^2 \frac{1}{2}d^T H^{(k)}d.$$

We use the Cauchy step to estimate the active set, and we define the EQP step, denoted by d_{qp}.

In practice, we may also try to take the LPEC step. If we are far from the solution, then this step may be acceptable to the filter. Because this step has already been computed, the additional cost in trying this step is negligible.

SLPEC-EQP Algorithm

Given (x_0, y_0), $\rho \in [\underline{\rho}, \overline{\rho}]$, set $k \leftarrow 0$; compute $\nabla f^{(k)}, \nabla c^{(k)}, \nabla F^{(k)}$

while *not optimal* **do**

 reset trust-region radius $\rho \in [\underline{\rho}, \overline{\rho}]$

 repeat

  solve LPEC$(z^{(k)}, \rho)$ for a first-order step d_{lp}

  if $\exists$ *solution d_{lp} of LPEC$(z^{(k)}, \mu)$* **then**

   if $d = 0$ **then terminate** B-stationary point found

   compute the Cauchy-step $d_c := \alpha_c d_{lp}$

   find the active sets $\mathcal{A}_c(z^{(k)} + d_c)$, $\mathcal{A}_y(z^{(k)} + d_c)$, and $\mathcal{A}_F(z^{(k)} + d_c)$.

   solve EQP$(z^{(k)} + d_c)$ and let the solution be d_{qp}

   compute predicted reduction Δq

   evaluate $f(z^{(k)} + d_{qp})$, $h_c(z^{(k)} + d_{qp})$, and $h_F(z^{(k)} + d_{qp})$

   if $z^{(k)} + d$ *acceptable to filter and* $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ **then**

    if $\Delta q < \delta (h^{(k)})^2$ **then**

     set $\rho_k = \rho$, $d^{(k)} = d_{qp}$, $\Delta q^{(k)} = \Delta q$, $\Delta f^{(k)} = \Delta f$

     add $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ to the filter   *h-type iteration*

    else if $\Delta f \geq \sigma \Delta q$ *and* $\Delta q \geq \delta (h^{(k)})^2$ **then**

     set $\rho_k = \rho$, $d^{(k)} = d$, $\Delta q^{(k)} = \Delta q$, $\Delta f^{(k)} = \Delta f$ *f-type iteration*

    else

     reduce trust-region radius $\rho = \rho/2$

   else

    reduce trust-region radius $\rho = \rho/2$

  else

   add $(h_c^{(k)}, h_F^{(k)}, f^{(k)})$ to filter

   enter **restoration phase** to find acceptable/compatible point, $z^{(k+1)}$

 until *new $z^{(k+1)}$ found*

 set $k = k + 1$, update gradients $\nabla f^{(k)}, \nabla c^{(k)}, \nabla F^{(k)}$ & test for convergence

5.2 Computational Considerations

We finish by providing some computational considerations. It should be clear that the algorithm closely resembles filter-SLQP methods that have been proposed by Chin and Fletcher (2003). This similarity is deliberate: it allows us to reuse the SLQP code (Leyffer, 2007) that we are currently developing to solve MPECs by simply replacing the step-computation through an LP by an LPEC.

At first sight, the SLPEC approach may appear computationally intractable because of the potential existence of 2^d LP-pieces that have to be solved at every iteration, where $d = |\mathcal{D}|$ is the number of degenerate indices. In many cases, however, we do not require the solution of all LP pieces. For example, if the current point is not stationary, then descend along *any* piece is sufficient to ensure convergence. Thus we do not need to solve the LPEC to global optimality, and instead we expect to be able to make progress by solving a small number of LPs at most.

In fact, if the current iterate satisfies an MPEC-LICQ, then it follows that there exists a common multiplier, and we can find a descend direction after a single LP solve. The same holds if the solution satisfies an MPEC-LICQ. In this case we can again compute a common multiplier and verify optimality after a single LP solve. Thus we do not believe that the computational burden of our method will become prohibitive.

6 Conclusions

We have presented a new algorithm for solving MPECs and have established global convergence under mild conditions. The algorithm solves an LPEC to predict the optimal active set and fixes the activities and solves an equality-constrained QP to accelerate local convergence. Global convergence is promoted through the use of a filter that distinguishes the general constraint infeasibility and the complementarity constraint violation.

We have also provided examples that show that commonly used stationarity concepts do not preclude the existence of trivial descend directions. These spurious stationarity concepts have been referred to as A-, C-, and M-stationarity. We have shown that popular methods such as NLP approaches and even SQPEC method are attracted to these spurious stationary points.

Our results can be extended in various ways. Clearly, we could replace the filter by a merit function, such as an ℓ_1 exact penalty function. Because we maintain complementarity, exactness results should be straightforward to establish. The proposed methods also extend to star-shaped optimization (Scholtes, 2004) and to optimization problems with vanishing constraints that arise for example in truss-topology design.

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