Optimization Problems with Equilibrium Constraints

GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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Outline

1. Introduction: Stackelberg Games

2. Difficulties with MPECs

3. Stationarity Conditions for MPECs
   - Bouligand and Strong Stationarity
   - Alphabet Soup of Spurious Stationarity
Introduction: Nash Games

ISO game

Nash producers

Nash Game: non-cooperative equilibrium of several producers

\[
\begin{align*}
  z_i^* &\in \arg\min_{z_i} b_i(\hat{z}) \\
  \text{subject to } & c_i(z_i) \geq 0 \\
  z_i &\geq 0
\end{align*}
\]

producer \( i \)

Producer \( i \) optimizes own \( z_i \), given other producers choices

- All producers \( \hat{z} = (z_1^*, \ldots, z_{i-1}^*, z_i, z_{i+1}^*, \ldots, z_l^*) \)
- No shared constraints (otherwise called Nash-Gournot)
- All producers/players are equal

Definition (Nash Equilibrium)

No producer \( i \) can improve objective, if other producer’s variables, \( z_j, \forall j \neq i \), remain unchanged.
Solution of Nash Games

Form first-order optimality conditions for each player ...

\[
\text{\begin{align*}
\text{(NCP)} \quad \begin{cases}
0 \leq \mu & \perp \nabla b(z) - \nabla c(z) \lambda \geq 0 \\
0 \leq \lambda & \perp c(z) \geq 0
\end{cases}
\end{align*}}
\]

where

- \( b(z) = (b_1(z), \ldots, b_k(z)) \) & \( c(z) = (c_1(z), \ldots, c_k(z)) \)
- \( \perp \) means \( \lambda^T c(z) = 0 \), either \( \lambda_i > 0 \) or \( c_i(z) > 0 \)
- Called a nonlinear complementarity problem (NCP)

Robust large scale solvers exist: e.g. PATH

Setting \( y = (z, \lambda, \mu)^T \) and \( F(y) = (b(z) - \nabla c(z) \lambda, c(z))^T \), we can rewrite (NCP) equivalently as

\[
0 \leq y \perp F(y) \geq 0
\]

... change of notation: \( y \) both variables and multipliers!
Stackelberg Games & Bilevel Optimization

Single dominant producer & Nash followers

\[
\begin{aligned}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) = 0 \\
& \quad 0 \leq y \perp F(x, y) \geq 0
\end{aligned}
\]

- Nash game \((0 \leq y \perp F(x, y) = 0)\)
  ... parameterized in leader's variables \(x\)
- Mathematical Program with Equilibrium Constraints (MPEC)
Bilevel Optimization as MPECs

Single dominant producer & Nash followers equivalent to

\[
\begin{aligned}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) = 0 \\
& \quad \min_y b(y) \\
& \quad \text{s.t. } d(y, x) \geq 0
\end{aligned}
\]

- Lower-level problem (min \( b(y) \) s.t. \( d(y, x) \geq 0 \))
  ... parameterized in leader's variables \( x \)
- Mathematical Program with Equilibrium Constraints (MPEC)
Example: Optimal Taxation Model

Government sets tax rates, \( t_g \), for certain goods to maximize revenue

- Consumers buy goods to maximize own utility function
- Consumers react to tax rates by changing purchase behavior
- Government is leader ... knows how consumers will react

Assume we have seven goods:

\[ G = \{ \text{Beer, Pizza, Movie, Wine, Cheese, Ballet, Leisure} \} \]

... and two classes of consumers

\[ C = \{ \text{Students, Professors} \} \]
Example: Optimal Taxation Model

Consumer $c$ buys quantities $q_{c,g} \geq 0$ of goods, $g \in \mathcal{G}$ to

$$\begin{align*}
\text{maximize} & \quad U_c(q) = \prod_{g \in \mathcal{G}} q_{c,g}^{\alpha_{c,g}} \\
\text{subject to} & \quad \sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c
\end{align*}$$

utility function

budget constraint

where $\sum \alpha_{c,g} = 1$, with prices, $p_g$, and tax-rates, $t_g$ of good $g \in \mathcal{G}$

KKT conditions of consumer $c$ are:

$$-\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in \mathcal{G}: g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c \quad \pi_c \geq 0 \quad \text{and} \quad 0 \leq q_{c,g} \perp \xi_{c,g} \geq 0$$
Example: Optimal Taxation Model

Government maximizes tax revenue subject to consumer actions

$$\max_t \sum_{c \in C} \sum_{g \in G} t_g q_{c,g} N_c$$

s.t. $$-\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in G : g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in G$$

$$\sum_{g \in G} p_g (1 + t_g) q_{c,g} \leq b_c \quad \perp \quad \pi_c \geq 0$$

$$0 \leq q_{c,g} \quad \perp \quad \xi_{c,g} \geq 0, \quad \forall c \in C, \forall g \in G$$

where $N_c$ is the number of consumers in class $c \in C$
Example: Optimal Taxation Model

Government maximizes tax revenue subject to consumer actions

$$\max_t \sum_{c \in C} \sum_{g \in G} t_g q_{c,g} N_c$$

s.t. $$-\alpha_{c,g} q_{c,g}^{(\alpha_{c,g} - 1)} \prod_{g' \in G: g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in G$$

$$\sum_{g \in G} p_g (1 + t_g) q_{c,g} \leq b_c \quad \pi_c \geq 0$$

$$0 \leq q_{c,g} \quad \xi_{c,g} \geq 0, \quad \forall c \in C, \forall g \in G$$

where $N_c$ is the number of consumers in class $c \in C$

So who gets taxed the most???
The Problem for the Rest of the Day

Mathematical Program with Equilibrium Constraints (MPEC)

\[
\begin{align*}
\minimize_{x,y} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad 0 \leq y \perp F(x, y) \geq 0
\end{align*}
\]

- \( f : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R} \), and \( c : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^m \) smooth
- Complementarity constraint: \( F : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^q \) smooth
  \( y_i = 0 \text{ or } F_i(x, y) = 0 \ldots \ y^T F(x, y) = 0 \)
- more general \( l \leq c(x, y) \leq u \): no problem
Stackelberg games [Stackelberg, 1952]

modeling of competition in deregulated electricity markets [Pieper, 2001, Hobbs et al., 2000]

volatility estimation in American option pricing [Huang and Pang, 1999]

transportation network design:

1. toll road pricing: how to set toll levels leader
2. consumers move optimally (Wardrop’s principle) followers

[Hearn and Ramana, 1997, Ferris et al., 1999]
MPEC: Engineering Applications

- design of structures involving friction [Ferris and Tin-Loi, 1999a]
- brittle fracture identification [Tin-Loi and Que, 2002]
- problems in elastoplasticity [Ferris and Tin-Loi, 1999b]
- process engineering models [Rico-Ramirez and Westerberg, 1999, Raghunathan and Biegler, 2002]
- floor planning (design of semi-conductors) [Anjos and Vanelli, 2002]
- obstacle problems (PDE); packaging problems [Outrata et al., 1998]
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Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad 0 \leq y \perp F(x, y) \geq 0
\end{align*}
\]

Equivalent smooth nonlinear program (NLP):

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad F(x, y) \geq 0 \quad \text{and} \quad y \geq 0 \\
& \quad y^T F(x, y) = 0
\end{align*}
\]

NLP solvers converge slowly, and sometimes fail completely!
Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

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\end{align*}
\]

NLP solvers converge slowly, and sometimes fail completely!
Example of Linear Convergence of SQP

Consider

\[
\min_{x,y} (x - 1)^2 + (y - 1)^2 \quad \text{subject to} \quad 0 \leq x \perp y \geq 0
\]

SQP method:

- **Start at** \((1, 1)\)

\[\begin{array}{c}
\text{... linear convergence to } (0, 0) \\
\text{... not even stationary!}
\end{array}\]

\[\begin{array}{c}
s = (0, 1) \\
s = (1, 0)
\end{array}\]
Example of Linear Convergence of SQP

Consider

\[
\begin{align*}
\text{minimize} & \quad (x - 1)^2 + (y - 1)^2 \\
\text{subject to} & \quad 0 \leq x \perp y \geq 0
\end{align*}
\]

SQP method:

- Start at \((1, 1)\)
- \((x_2, y_2) = (1/2, 1/2)\)
Example of Linear Convergence of SQP

Consider

\[
\text{minimize } (x - 1)^2 + (y - 1)^2 \quad \text{subject to } 0 \leq x \perp y \geq 0
\]

SQP method:

- Start at \((1, 1)\)
- \((x_2, y_2) = (1/2, 1/2)\)
- \((x_3, y_3) = (1/2^k, 1/2^k)\)

... linear convergence to \((0, 0)\)
... multipliers \(\to \infty\)

... not even stationary! \(s = (0, 1)\) \(s = (1, 0)\) descend!
A Nonlinear Programming Approach

Replace equilibrium \(0 \leq x_1 \perp x_2 \geq 0\) by \(X_1 x_2 \leq 0\) or \(x_1^T x_2 \leq 0\)

\[\Rightarrow\] standard nonlinear program (NLP)

\[
\begin{align*}
& \text{minimize } f(x) \\
& \text{subject to } c(x) \geq 0 \\
& x_1, x_2 \geq 0 \\
& X_1 x_2 \leq 0
\end{align*}
\]

Advantage: standard (?) NLP; use large-scale solvers ...

Snag: nonlinear program (NLP) violates standard assumptions!
Mangasarian Fromowitz CQ fails

Mangasarian Fromowitz Constraint Qualification at feasible \( \hat{x} \):

\[
\hat{x}_1 = 0, \quad \hat{x}_2 > 0
\]

\[\Rightarrow x_1 \geq 0, \quad \text{and} \quad x_2 x_1 \leq 0 \quad \text{active}\]

\[\Rightarrow \text{MFCQ: } s_1 > 0, \quad \text{and} \quad \hat{x}_2 s_1 < 0\]

MFCQ is important (minimalist) stability assumption for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set unbounded ... \( \nabla^2 \mathcal{L} \) may blow up
2. Constraint gradients linearly dependent ... ill-conditioned steps
3. Central path does not exist ... IPMs may not work at all!
Consider the two QPECs

\[
\begin{align*}
\text{minimize} & \quad f_i(x, y) \\
\text{subject to} & \quad 0 \leq y \perp y - x \geq 0
\end{align*}
\]

with \( f_1(z) = (x - 1)^2 + y^2 \) and \( f_2(z) = x^2 + (y - 1)^2 \)

Solution at \((x, y)^* = (1/2, 1/2)^T\)
Dependent Constraints and Unbounded Multiplier Sets

Equivalent NLP of QPECs is

\[
\begin{cases}
\text{minimize } & f_i(z) \\
\text{subject to } & y \geq 0 \\
& \nu \geq 0 \\
& y - x \geq 0 \\
& \lambda \geq 0 \\
& y (y - x) \leq 0 \\
& \xi \geq 0.
\end{cases}
\]

with KKT conditions:

\[
\begin{pmatrix}
-1 \\
1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
= \lambda^* \begin{pmatrix}
-1 \\
1
\end{pmatrix} - \xi^* \begin{pmatrix}
-\frac{1}{2} \\
\frac{1}{2}
\end{pmatrix}.
\]

… active constraint normals are clearly dependent!
Dependent Constraints and Unbounded Multiplier Sets

Since \( y^* = \frac{1}{2} > 0 \) we see \( \nu^* = 0 \), and multiplier sets ...

\[
\mathcal{M}_1 = \{(\lambda, \xi) \mid \xi \geq 0, \lambda + \frac{1}{2}\xi = 1\}
\]

\[
\mathcal{M}_2 = \{(\lambda, \xi) \mid \lambda \geq 0, -\lambda + \frac{1}{2}\xi = 1\},
\]

... are unbounded
Inconsistent Linearizations

MPECs can have inconsistent linearizations arbitrarily close to stationary point

\[
\begin{align*}
\min_{z} & \quad x + y \\
\text{subject to} & \quad y^2 \geq 1 \\
& \quad 0 \leq x \perp y \geq 0.
\end{align*}
\]

Nice solution: \((x, y)^* = (0, 1)^T\) multipliers \(\lambda^* = 0.5\)

Linearize at \((\hat{x}, \hat{y}) = (\epsilon, 1 - \delta)^T\) with \(\epsilon, \delta > 0\):

\[
(1 - \delta)^2 + 2(1 - \delta)(y - (1 - \delta)) \geq 1 \quad \Rightarrow \quad y \geq \frac{1 + (1 - \delta)^2}{2(1 - \delta)} > 1
\]

and

\[
(1 - \delta)\epsilon + (1 - \delta)(x - \epsilon) + \epsilon(y - (1 - \delta)) \leq 0 \quad \Rightarrow \quad y \leq 1 - \delta < 1
\]
How Else Can We Solve MPECs?

\[
\begin{aligned}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad F(x, y) \geq 0 \quad \text{and} \quad y \geq 0 \\
& \quad y^T F(x, y) = 0
\end{aligned}
\]

Goal

Want to use the good NLP solvers, such as IPM, SQP, SLQP, ...

Trouble caused by too many dependent active constraints:
\( F(x, y) = 0 \) and \( y = 0 \) and \( y^T F(x, y) = 0 \) ... remove one!

Two alternative approaches that use NLP solvers:

1. Relax the complementarity constraint
2. Penalize the complementarity constraint
NLP-Based Relaxation Approach to MPECs

Formulate a relaxed NLP

\[(R\text{-NLP}(\rho))\]

\[
\begin{aligned}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad F(x, y) \geq 0 \quad \text{and} \quad y \geq 0 \\
& \quad y^T F(x, y) = \rho
\end{aligned}
\]

... for \(\rho \downarrow 0\)

Given initial \(\rho > 0\)

repeat
  Solve \((R\text{-NLP}(\rho))\) for \((x^\rho, y^\rho)\)
  Reduce \(\rho := \rho / 4\)
until \((x^\rho, y^\rho)\) is solution of MPEC;
NLP-Based Penalization Approach to MPECs

Formulate a penalized NLP

\[
\begin{align*}
\text{minimize} & \quad f(x, y) + \pi \| y^T F(x, y) \| \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad F(x, y) \geq 0 \quad \text{and} \quad y \geq 0
\end{align*}
\]

(P-NLP(\(\rho\)))

... for \(\pi \uparrow 0\) ... problem satisfies MFCQ!

Given initial \(\pi > 0\)

repeat

| Solve (P-NLP(\(\pi\))) for \((x^\pi, y^\pi)\) |
| Reduce \(\pi := 4\pi\) |

until \((x^\pi, y^\pi) is solution of MPEC;\)

Relaxation and penalization are loosely related ...
An Even Simpler Trick Seems to Work

Consider an alternative (lazy) reformulation of MPEC

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad 0 \leq y \perp F(x, y) \geq 0
\end{align*}
\]

Introduce slack variables \( s \):

- Write \( F(x, y) = s \) as nonlinear equation
- Simplify the complementarity to bilinear inequality \( y^T s \leq 0 \)
- Equivalent, because \( s, y \geq 0 \) ... solvers satisfy bounds easily

Equivalent smooth nonlinear program (NLP):

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad c(x, y) \geq 0 \\
& \quad F(x, y) = s, \quad s \geq 0, \quad y \geq 0 \quad \text{and} \quad y^T s \leq 0
\end{align*}
\]

... more in the next lecture!
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MPEC Bouligand-Stationarity

Definition (MPEC B-Stationarity)

$(x^*, y^*)$ is \textit{B-stationary}, iff $d = 0$ solves LPEC

\[
\begin{align*}
\text{minimize} \quad & g^*^T d \\
\text{subject to} \quad & c^* + A^*^T d \geq 0, \\
& 0 \leq y^* + d_y \perp F^* + B^*^T d \geq 0,
\end{align*}
\]

where $g^* = \nabla f(x^*, y^*)$, $A^* = \nabla c(x^*, y^*)$, $B^* = \nabla F(x^*, y^*)$

B-stationarity is a structural stationarity condition

- Applies stationarity to nonlinear functions
- Retains structure of the problem $\Rightarrow$ strong result
- Absence of feasible descend directions!

... similar to LP being stationary for NLP
MPEC Strong-Stationarity

- \((x^*, y^*)\) is weakly-stationary, iff \(\exists \lambda, \mu, \) and \(\nu:\)

\[
g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} = 0, \\
0 \leq c^* \perp \lambda \geq 0, \\
0 \leq y^* \perp F^* \geq 0.
\]

where \(\nu \perp y^*\) and \(\mu \perp F(x, y)\) ... \(\mu, \nu\) unrestricted

- Degenerate complementarity conditions:

\[
D(z) := \{ i : y_i = 0 = F_i(z) \}
\]

- \((x^*, y^*)\) is strongly-stationary iff

\[
\mu_i \geq 0, \ \nu_i \geq 0, \ \forall i \in D^*
\]

... equivalent to KKT conditions of equivalent NLP
Alphabet Soup of Spurious Stationarity

$(x^*, y^*)$ is weakly-stationary, iff $\exists \lambda, \mu, \text{ and } \nu$:

$$g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} = 0,$$

$$0 \leq c^* \perp \lambda \geq 0,$$

$$0 \leq y^* \perp F^* \geq 0.$$

where $\nu \perp y^*$ and $\mu \perp F(x, y)$

Degenerate complementarity: $D(z) := \{ i : y_i = 0 = F_i(z) \}$

- A-stationary, iff $\mu_i \geq 0 \text{ or } \nu_i \geq 0, \forall i \in D^*$
- C-stationary, iff $\mu_i \nu_i \geq 0 \forall i \in D^*$
- M-stationary, iff $(\mu_i > 0 \text{ and } \nu_i > 0) \text{ or } \mu_i \nu_i = 0, \forall i \in D^*$

all have trivial descend directions
Spuriousness of C-Stationarity

Consider \( \min (x - 1)^2 + (y - 1)^2 \) subject to \( 0 \leq y \perp x \geq 0 \):

\[(0, 0) \text{ C-stationary: } \mu = \nu = -2 < 0!!! \Rightarrow \exists \text{ descend directions} \]
Spuriousness of A/M-Stationarity

Consider $\min (x - 1)^2 + y^3 + y^2$ subject to $0 \leq y \perp x \geq 0$

$(0, 0)$ M/A-stationarity: $\mu = 0, \nu = -2 < 0!!$

$\Rightarrow$ exists descend directions
Alphabet Soup of Stationarity

A/B/C/M/S-stationarity equivalent, iff $\mathcal{D}^* = \emptyset$
What Have We Learned?

Complementarity constraints are important class of problems
- Arise in many applications ... useful modeling paradigm
- Students should pay more taxes than their professors

MPECs are a challenging class of problems
- Violate MFCQ $\Rightarrow$ unbounded multipliers, infeasible linearizations
- NLP solvers can fail

Extended optimality conditions
- B-stationarity is the best ... and most difficult
- Strong stationarity is good ... but does not always hold
- Many useless stationarity concepts: A-, C-, L-, M-, W- ...


Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints. 

On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints.

Engineering and economic applications of complementarity problems.

Local convergence of SQP methods for mathematical programs with equilibrium constraints.
Numerical Analysis Report NA/209, Department of Mathematics, University of Dundee, Dundee, DD1 4HN, UK.

Hearn, D. and Ramana, M. (1997(?)).
Solving congestion toll pricing models.

Strategic gaming analysis for electric power systems: An mpec approach.

A mathematical programming with equilibrium approach to the implied volatility surface of American options.
Technical report, Dept. of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21218-2682, USA.

Generalized stationary points and an interior point method for mathematical programs with equilibrium constraints.
Preprint, Singapore, School of Business.

Mathematical Programs with Equilibrium Constraints.
Cambridge University Press.

Nonsmooth Approach to Optimization Problems with Equilibrium Constraints.

Quasi-Variational Inequalities, Generalized Nash Equilibria, and Multi-Leader-Follower Games.
Preprint The Johns Hopkins Universit, Baltimore, USA.

Algorithms for Mathematical Programs with Equilibrium Constraints with Applications to deregulated electricity markets.
PhD thesis, Department of Management Science and Engineering, Stanford University.


