Optimization Problems with Equilibrium Constraints

GIAN Short Course on Optimization:
Applications, Algorithms, and Computation

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Outline

Introduction: Nash Games

Nash Game: non-cooperative equilibrium of several producers

$$z_i^* \in \begin{cases} \underset{z_i}{\text{argmin}} & b_i(\hat{z}) \\ \text{subject to} & c_i(z_i) \geq 0 \\ & z_i \geq 0 \end{cases} \quad \text{producer } i$$

Producer i optimizes own z_i, given other producers choices

- All producers $\hat{z} = (z_1^*, \ldots, z_{i-1}^*, z_i, z_{i+1}^*, \ldots, z_I^*)$
- No shared constraints (otherwise called Nash-Gournot)
- All producers/players are equal

Definition (Nash Equilibrium)

No producer i can improve objective, if other producer's variables, $z_i, \forall j \neq i$, remain unchanged.

Solution of Nash Games

Form first-order optimality conditions for each player ...

$$\text{(NCP)} \qquad \begin{cases} 0 \le \mu \perp \nabla b(z) - \nabla c(z)\lambda \ge 0 \\ 0 \le \lambda \perp c(z) \ge 0 \end{cases}$$

where

- $b(z) = (b_1(z), \ldots, b_k(z))$ & $c(z) = (c_1(z), \ldots, c_k(z))$
- $\perp$ means $\lambda^T c(z) = 0$, either $\lambda_i > 0$ or $c_i(z) > 0$
- Called a nonlinear complementarity problem (NCP)
- Robust large scale solvers exist: e.g. `PATH`

Setting $y = (z, \lambda, \mu)^T$ and $F(y) = (b(z) - \nabla c(z)\lambda,\ c(z))^T$, we can rewrite (NCP) equivalently as

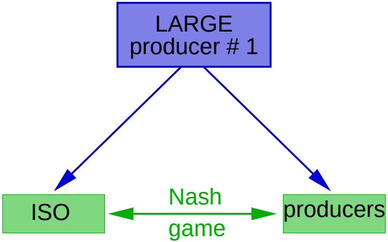
$$0 \le y \ \perp\ F(y) \ge 0$$

... change of notation: y both variables and multipliers!

Stackelberg Games & Bilevel Optimization

Single dominant producer & Nash followers

$$\left\{ \begin{array}{ll} \underset{x \geq 0, y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) = 0 \\ & 0 \leq y \ \perp \ F(x,y) \geq 0 \end{array} \right.$$

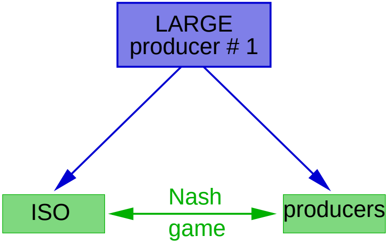


- Nash game $(0 \leq y \ \perp \ F(x,y) = 0)$
  ... parameterized in leader's variables x
- Mathematical Program with Equilibrium Constraints (MPEC)

Bilevel Optimization as MPECs

Single dominant producer & Nash followers equivalent to

$$\begin{cases} \underset{x \geq 0, y}{\text{minimize}} \quad f(x,y) \\ \text{subject to } c(x,y) = 0 \\ \qquad\qquad\qquad \begin{cases} \min\limits_{y} \; b(y) \\ \text{s.t. } \; d(y,x) \geq 0 \end{cases} \end{cases}$$



- Lower-level problem ($\min \; b(y)$ s.t. $d(y,x) \geq 0$)
  ... parameterized in leader's variables x
- Mathematical Program with Equilibrium Constraints (MPEC)

Example: Optimal Taxation Model

Government sets tax rates, t_g, for certain goods to maximize revenue

- Consumers buy goods to maximize own utility function
- Consumers react to tax rates by changing purchase behavior
- Government is leader ... knows how consumers will react

Assume we have seven goods:

$$\mathcal{G} = \{\text{Beer, Pizza, Movie, Wine, Cheese, Ballet, Leisure}\}$$

... and two classes of consumers

$$\mathcal{C} = \{\text{Students, Professors}\}$$

Example: Optimal Taxation Model

Consumer c buys quantities $q_{c,g} \geq 0$ of goods, $g \in \mathcal{G}$ to

$$\begin{cases} \underset{q}{\text{maximize}} & U_c(q) = \prod_{g \in \mathcal{G}} q_{c,g}^{\alpha_{c,g}} & \text{utility function} \\ \text{subject to} & \sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c & \text{budget constraint} \end{cases}$$

where $\sum \alpha_{c,g} = 1$, with prices, p_g, and tax-rates, t_g of good $g \in \mathcal{G}$

KKT conditions of consumer c are:

$$-\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in \mathcal{G} : g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c \perp \pi_c \geq 0 \quad \text{and} \quad 0 \leq q_{c,g} \perp \xi_{c,g} \geq 0$$

Example: Optimal Taxation Model

Government maximizes tax revenue subject to consumer actions

$$\max_{t} \sum_{c \in \mathcal{C}} \sum_{g \in \mathcal{G}} t_g q_{c,g} N_c$$

$$\text{s.t.} \quad -\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g' \in \mathcal{G} : g' \neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g (1 + t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g \in \mathcal{G}} p_g (1 + t_g) q_{c,g} \leq b_c \ \perp \ \pi_c \geq 0$$

$$0 \leq q_{c,g} \ \perp \ \xi_{c,g} \geq 0, \qquad \forall c \in \mathcal{C},\ \forall g \in \mathcal{G}$$

where N_c is the number of consumers in class $c \in \mathcal{C}$

Example: Optimal Taxation Model

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$$\text{s.t.} \quad -\alpha_{c,g} q_{c,g}^{(\alpha_{c,g}-1)} \prod_{g'\in\mathcal{G}:g'\neq g} q_{c,g'}^{\alpha_{c,g'}} + \pi_c p_g(1+t_g) - \xi_{c,g} = 0 \quad \forall g \in \mathcal{G}$$

$$\sum_{g\in\mathcal{G}} p_g(1+t_g) q_{c,g} \leq b_c \;\perp\; \pi_c \geq 0$$

$$0 \leq q_{c,g} \;\perp\; \xi_{c,g} \geq 0, \qquad \forall c \in \mathcal{C},\; \forall g \in \mathcal{G}$$

where N_c is the number of consumers in class $c \in \mathcal{C}$

So who gets taxed the most???

The Problem for the Rest of the Day

Mathematical Program with Equilibrium Constraints (MPEC)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \perp F(x,y) \geq 0 \end{cases}$$

- $f : R^p \times R^q \rightarrow R$, and $c : R^p \times R^q \rightarrow R^m$ smooth
- Complementarity constraint: $F : R^p \times R^q \rightarrow R^q$ smooth
  $y_i = 0$ or $F_i(x,y) = 0$ ... $y^T F(x,y) = 0$
- more general $l \leq c(x,y) \leq u$: no problem

MPEC: Economic Applications

- Stackelberg games [Stackelberg, 1952]
- modeling of competition in deregulated electricity markets [Pieper, 2001, Hobbs et al., 2000]
- volatility estimation in American option pricing [Huang and Pang, 1999]
- transportation network design:
  1. toll road pricing: how to set toll levels — leader
  2. consumers move optimally (Wardrop's principle) — followers

  [Hearn and Ramana, 1997, Ferris et al., 1999]

MPEC: Engineering Applications

- design of structures involving friction [Ferris and Tin-Loi, 1999a]
- brittle fracture identification [Tin-Loi and Que, 2002]
- problems in elastoplasticity [Ferris and Tin-Loi, 1999b]
- process engineering models [Rico-Ramirez and Westerberg, 1999, Raghunathan and Biegler, 2002]
- floor planning (design of semi-conductors) [Anjos and Vanelli, 2002]
- obstacle problems (PDE); packaging problems [Outrata et al., 1998]

Outline

Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \;\perp\; F(x,y) \geq 0 \end{cases}$$

Equivalent smooth nonlinear program (NLP):

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = 0 \end{cases}$$

Why Not Simply Solve MPECs as NLPs?

Mathematical Program with Equilibrium Constraints (MPEC)

$$
\begin{cases}
\underset{x,y}{\text{minimize}} & f(x,y) \\
\text{subject to} & c(x,y) \geq 0 \\
& 0 \leq y \;\perp\; F(x,y) \geq 0
\end{cases}
$$

Equivalent smooth nonlinear program (NLP):

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\underset{x,y}{\text{minimize}} & f(x,y) \\
\text{subject to} & c(x,y) \geq 0 \\
& F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\
& y^T F(x,y) = 0
\end{cases}
$$

NLP solvers converge slowly, and sometimes fail completely!

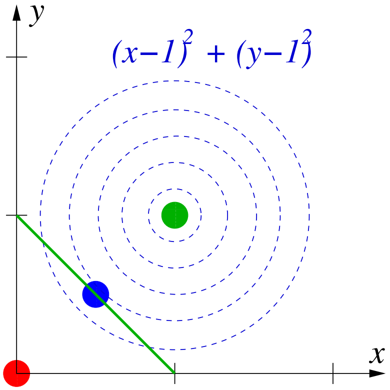
Example of Linear Convergence of SQP

Consider

$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \perp y \ge 0$$

SQP method:

- Start at $(1,1)$



Example of Linear Convergence of SQP

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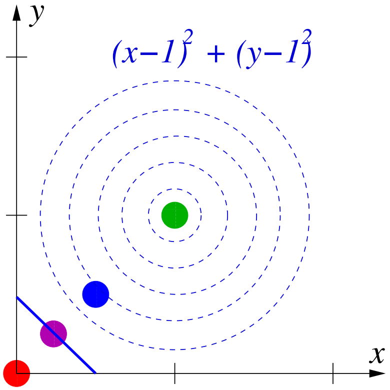
$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \ \perp \ y \ge 0$$

SQP method:

- Start at $(1,1)$
- $(x_2, y_2) = (1/2,\ 1/2)$



Example of Linear Convergence of SQP

Consider

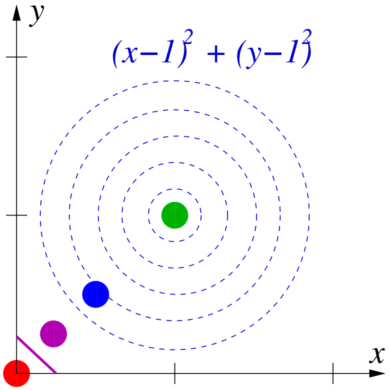
$$\underset{x,y}{\text{minimize}}\ (x-1)^2 + (y-1)^2 \quad \text{subject to} \quad 0 \le x \perp y \ge 0$$

SQP method:

- Start at $(1,1)$
- $(x_2, y_2) = (1/2,\ 1/2)$
- $(x_3, y_3) = (1/2^k,\ 1/2^k)$

... linear convergence to $(0,0)$

... multipliers $\to \infty$



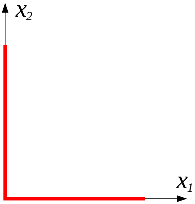
... not even stationary! $s = (0,1)$ $s = (1,0)$ descend!

A Nonlinear Programming Approach

Replace equilibrium $0 \le x_1 \perp x_2 \ge 0$ by $X_1 x_2 \le 0$ or $x_1^T x_2 \le 0$

$\Rightarrow$ standard nonlinear program (NLP)

$$(\text{NLP}) \begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \ge 0 \\ & x_1, \; x_2 \ge 0 \\ & \boxed{X_1 x_2 \le 0} \end{cases}$$



Advantage: standard (?) NLP; use large-scale solvers ...
Snag: nonlinear program (NLP) violates standard assumptions!

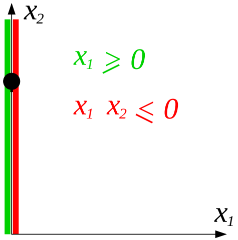
Mangasarian Fromowitz CQ fails

Mangasarian Fromowitz Constraint Qualification at feasible $\hat{x}$:

$$\hat{x}_1 = 0,\ \hat{x}_2 > 0$$

$$\Rightarrow x_1 \geq 0, \quad \text{and} \quad x_2 x_1 \leq 0 \quad \text{active}$$

$$\Rightarrow \text{MFCQ: } s_1 > 0, \quad \text{and } \hat{x}_2 s_1 < 0$$



MFCQ is important (minimalist) stability assumption for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set unbounded ... $\nabla^2 \mathcal{L}$ may blow up
2. Constraint gradients linearly dependent ... ill-conditioned steps
3. Central path does not exist ... IPMs may not work at all!

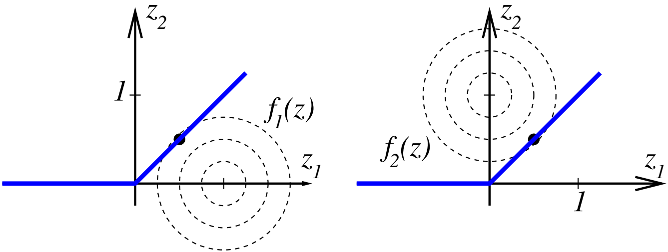
Dependent Constraints and Unbounded Multiplier Sets

Consider the two QPECs

$$\begin{cases} \underset{z}{\text{minimize}} & f_i(x,y) \\ \text{subject to } 0 \leq y \ \perp \ y - x \geq 0 \end{cases}$$

with $f_1(z) = (x-1)^2 + y^2$ and $f_2(z) = x^2 + (y-1)^2$

Solution at $(x,y)^* = (1/2, 1/2)^T$



Dependent Constraints and Unbounded Multiplier Sets

Equivalent NLP of QPECs is

$$\begin{cases} \underset{z}{\text{minimize}} & f_i(z) & \text{multiplier} \\ \text{subject to} & y \geq 0 & \nu \geq 0 \\ & y - x \geq 0 & \lambda \geq 0 \\ & y\,(y-x) \leq 0 & \xi \geq 0. \end{cases}$$

with KKT conditions:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda^* \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \xi^* \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

... active constraint normals are clearly dependent!

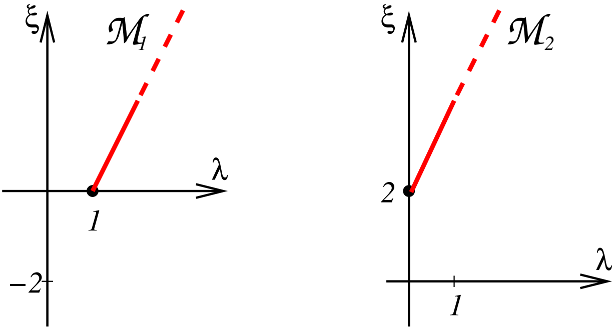
Dependent Constraints and Unbounded Multiplier Sets

Since $y^* = \frac{1}{2} > 0$ we see $\nu^* = 0$, and multiplier sets ...

$$\mathcal{M}_1 \; = \; \left\{ (\lambda, \xi) \mid \xi \geq 0, \; \lambda + \tfrac{1}{2}\xi = 1 \right\}$$

$$\mathcal{M}_2 \; = \; \left\{ (\lambda, \xi) \mid \lambda \geq 0, \; -\lambda + \tfrac{1}{2}\xi = 1 \right\},$$

... are unbounded



Inconsistent Linearizations

MPECs can have inconsistent linearizations arbitrarily close to stationary point

$$\begin{cases} \underset{z}{\text{minimize}} & x + y \\ \text{subject to} & y^2 \geq 1 \\ & 0 \leq x \perp y \geq 0. \end{cases}$$

Nice solution: $(x, y)^* = (0, 1)^T$ multipliers $\lambda^* = 0.5$
Linearize at $(\hat{x}, \hat{y}) = (\epsilon, 1 - \delta)^T$ with $\epsilon, \delta > 0$:

$$(1-\delta)^2 + 2(1-\delta)(y - (1-\delta)) \geq 1 \quad \Rightarrow \quad y \geq \frac{1 + (1-\delta)^2}{2(1-\delta)} > 1$$

and

$$(1-\delta)\epsilon + (1-\delta)(x - \epsilon) + \epsilon(y - (1-\delta)) \leq 0 \quad \Rightarrow \quad y \leq 1 - \delta < 1$$

How Else Can We Solve MPECs?

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = 0 \end{cases}$$

Goal

Want to use the good NLP solvers, such as IPM, SQP, SLQP, ...
Trouble caused by too many dependent active constraints:
$F(x,y) = 0$ and $y = 0$ and $y^T F(x,y) = 0$ ... remove one!

Two alternative approaches that use NLP solvers:

1. Relax the complementarity constraint
2. Penalize the complementarity constraint

NLP-Based Relaxation Approach to MPECs

Formulate a relaxed NLP

$$(\text{R-NLP}(\rho)) \qquad \begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) \geq 0 \quad \text{and} \quad y \geq 0 \\ & y^T F(x,y) = \rho \end{cases}$$

... for $\rho \searrow 0$

Given initial $\rho > 0$
repeat
  Solve (R-NLP(ρ)) for (x^ρ, y^ρ)
  Reduce $\rho := \rho/4$
until (x^ρ, y^ρ) *is solution of MPEC*;

NLP-Based Penalization Approach to MPECs

Formulate a penalized NLP

$$(\text{P-NLP}(\rho)) \qquad \begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) + \pi \| y^T F(x,y) \| \\ \text{subject to } c(x,y) \ge 0 & \\ \qquad\qquad F(x,y) \ge 0 \quad \text{and} \quad y \ge 0 \end{cases}$$

... for $\pi \nearrow 0$ ... problem satisfies MFCQ!

Given initial $\pi > 0$

repeat

  Solve (P-NLP(π)) for (x^π, y^π)

  Reduce $\pi := 4\pi$

until (x^π, y^π) *is solution of MPEC*;

Relaxation and penalization are loosely related ...

An Even Simpler Trick Seems to Work

Consider an alternative (lazy) reformulation of MPEC

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & 0 \leq y \;\perp\; F(x,y) \geq 0 \end{cases}$$

Introduce slack variables s:

- Write $F(x,y) = s$ as nonlinear equation
- Simplify the complementarity to bilinear inequality $y^T s \leq 0$
- Equivalent, because $s, y \geq 0$ ... solvers satisfy bounds easily

Equivalent smooth nonlinear program (NLP):

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & c(x,y) \geq 0 \\ & F(x,y) = s, \quad s \geq 0, \quad y \geq 0 \quad \text{and} \quad y^T s \leq 0 \end{cases}$$

... more in the next lecture!

Outline

MPEC Bouligand-Stationarity

Definition (MPEC B-Stationarity)

(x^*, y^*) is *B-stationary*, iff $d = 0$ solves LPEC

$$
\begin{aligned}
\underset{d}{\text{minimize}} \quad & {g^*}^T d \\
\text{subject to } \; & c^* + {A^*}^T d \ge 0, \\
& 0 \le y^* + d_y \;\perp\; F^* + {B^*}^T d \ge 0,
\end{aligned}
$$

where $g^* = \nabla f(x^*, y^*)$, $\quad A^* = \nabla c(x^*, y^*)$, $\quad B^* = \nabla F(x^*, y^*)$

B-stationarity is a structural stationarity condition

- Applies stationarity to nonlinear functions
- Retains structure of the problem $\Rightarrow$ strong result
- Absence of feasible descend directions!

  ... similar to LP being stationary for NLP

MPEC Strong-Stationarity

- (x^*, y^*) is weakly-stationary, iff $\exists\ \lambda$, μ, and ν:

$$\begin{aligned} g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} &= 0, \\ 0 \leq c^* \ \perp \ \lambda &\geq 0, \\ 0 \leq y^* \ \perp \ F^* &\geq 0. \end{aligned}$$

  where $\nu \ \perp \ y^*$ and $\mu \ \perp \ F(x, y)$ ... μ, ν unrestricted
- Degenerate complementarity conditions:

$$\mathcal{D}(z) := \{i : y_i = 0 = F_i(z)\}$$

- (x^*, y^*) is strongly-stationary iff

$$\mu_i \geq 0, \ \nu_i \geq 0, \ \forall i \in \mathcal{D}^*$$

  ... equivalent to KKT conditions of equivalent NLP

Alphabet Soup of Spurious Stationarity

(x^*, y^*) is weakly-stationary, iff $\exists\ \lambda$, μ, and ν:

$$
\begin{aligned}
g^* - A^*\lambda - B^*\mu - \begin{pmatrix} 0 \\ \nu \end{pmatrix} &= 0, \\
0 \le c^* \ \perp \ \lambda &\ge 0, \\
0 \le y^* \ \perp \ F^* &\ge 0.
\end{aligned}
$$

where $\nu \ \perp \ y^*$ and $\mu \ \perp \ F(x,y)$

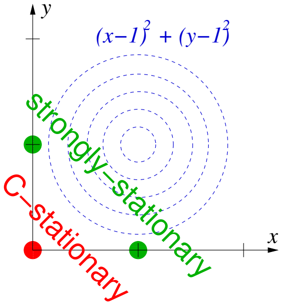
Degenerate complementarity: $\mathcal{D}(z) := \left\{ i : y_i = 0 = F_i(z) \right\}$

- A-stationary, iff $\mu_i \ge 0$ or $\nu_i \ge 0, \ \forall i \in \mathcal{D}^*$
- C-stationary, iff $\mu_i \nu_i \ge 0 \ \forall i \in \mathcal{D}^*$
- M-stationary, iff $\left(\mu_i > 0 \text{ and } \nu_i > 0\right)$ or $\mu_i \nu_i = 0, \ \forall i \in \mathcal{D}^*$

all have trivial descend directions

Spuriousness of C-Stationarity

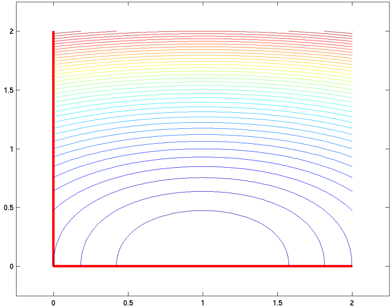
Consider min $(x-1)^2 + (y-1)^2$ subject to $0 \leq y \perp x \geq 0$:



$(0,0)$ C-stationary: $\mu = \nu = -2 < 0$!!!
$\Rightarrow \exists$ descend directions

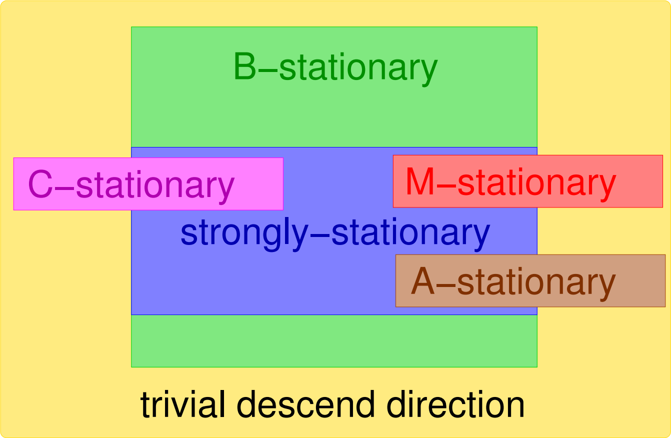
Spuriousness of A/M-Stationarity

Consider min $(x-1)^2 + y^3 + y^2$ subject to $0 \leq y \perp x \geq 0$



(0, 0) M/A-stationarity: $\mu = 0, \nu = -2 < 0$!!!
$\Rightarrow$ *exists* descend directions

Alphabet Soup of Stationarity



A/B/C/M/S-stationarity equivalent, iff $\mathcal{D}^* = \emptyset$

What Have We Learned?

Complementarity constraints are important class of problems

- Arise in many applications ... useful modeling paradigm
- Students should pay more taxes than their professors

MPECs are a challenging class of problems

- Violate MFCQ $\Rightarrow$ unbounded multipliers, infeasible linearizations
- NLP solvers can fail

Extended optimality conditions

- B-stationarity is the best ... and most difficult
- Strong stationarity is good ... but does not always hold
- Many useless stationarity concepts: A-, C-, L-, M-, W- ...

Anitescu, M. (2000).
On solving mathematical programs with complementarity constraints as nonlinear programs.
Preprint ANL/MCS-P864-1200, MCS Division, Argonne National Laboratory, Argonne, IL, USA.

Anjos, M. and Vanelli, A. (2002).
A new mathematical programming framework for facility layout problems.
Technical Report UW-E&CE#2002-04,, Department of Electrical & Computer Engineering, University of Waterloo,, Canada,.

Bard, J. (1988).
Convex two-level optimization.
Mathematical Programming, 40(1):15–27.

Benson, H., Shanno, D. F. and Vanderbei, R. V. D. (2003).
LOQO: An Interior-Point Methods for Nonconvex Nonlinear Programming.
Talk at ISMP-2003.

DeMiguel, V., Friedlander, M.P., Nogales, F.J. and Scholtes, S. (2003).
A superlinearly convergent interior point method for MPECs.
tALK AT ISMP-2003.

Ferris, M., Meeraus, A., and Rutherford, T. (1999).
Computing Wardropian equilibrium in a complementarity framework.
Optimization Methods and Software, 10:669–685.

Ferris, M. and Tin-Loi, F. (1999a).

Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints.
Mathematical Programming Technical Report 99-01, University of Wisconsin.

Ferris, M. and Tin-Loi, F. (1999b).
On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints.
Computer Methods in Applied Mechanics and Engineering, 174:107–120.

Ferris, M. and Pang, J. (1997).
Engineering and economic applications of complementarity problems.
SIAM Review, 39(4):669–713.

Fletcher, R., Leyffer, S., Ralph, D., and Scholtes, S. (2002).
Local convergence of SQP methods for mathematical programs with equilibrium constraints.
Numerical Analysis Report NA/209, Department of Mathematics, University of Dundee, Dundee, DD1 4HN, UK.

Hearn, D. and Ramana, M. (1997(?)).
Solving congestion toll pricing models.
Technical report, Department of Industrial and Systems Engineering, University of Florida, `http://www.ise.ufl.edu/hearn/crt.ps`.

Hobbs, B., Metzler, C., and Pang, J.-S. (2000).
Strategic gaming analysis for electric power systems: An mpec approach.
IEEE Transactions on Power Systems, 15(2):638–645.

Huang, J. and Pang, J.-S. (1999).

A mathematical programming with equilibrium approach to the implied volatility surface of American options.
Technical report, Dept. of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21218-2682, USA.

Liu, X. and Sun, J. (2002).
Generalized stationary points and an interior point method for mathematical programs with equilibrium constraints.
Preprint, Singapore, School of Business.

Luo, Z.-Q., Pang, J.-S., and Ralph, D. (1996).
Mathematical Programs with Equilibrium Constraints.
Cambridge University Press.

Outrata, J., Kocvara, M., and Zowe, J. (1998).
Nonsmooth Approach to Optimization Problems with Equilibrium Constraints.
Kluwer Academic Publishers, Dordrecht.

J.-S. Pang and M. Fukushima (2002).
Quasi-Variational Inequalities, Generalized Nash Equilibria, and Multi-Leader-Follower Games.
Preprint The Johns Hopkins Universit, Baltimore, USA.

Pieper, H. (2001).
Algorithms for Mathematical Programs with Equilibrium Constraints with Applications to deregulated electricity markets.
PhD thesis, Department of Management Science and Engineering, Stanford University.

Raghunathan, A. and Biegler, L. T. (2002).
MPEC formulations and algorithms in process engineering.
Technical report, CMU Chemical Engineering.

Rico-Ramirez, V. and Westerberg, A. (1999).
Conditional modeling. 2. solving using complementarity and boundary-crossing formulations.
Industrial Engineering and Chemistry Research, 38:531–553.

Scholtes, S. (2001).
Convergence properties of regularization schemes for mathematical programs with complementarity constraints.
SIAM Journal on Optimization, 11(4):918–936.

Stackelberg, H. V. (1952).
The Theory of Market Economy.
Oxford University Press.

Tin-Loi, F. and Que, N. (2002).
Nonlinear programming approaches for an inverse problem in quasibrittle fracture.
Mechanical Sciences, 44:843–858.

Wilmott, P., Dewyne, J., and Howison, S. (1993).
Option Pricing: Mathematical Models and Computation.
Oxford Financial Press.