Optimization of Dynamical Systems 2018: MINLP Exercises

1. Consider the gear-train design problem for best matching gear ratio

$$\underset{x}{\text{minimize}} \left(\frac{1}{6.931} - \frac{x_3 x_2}{x_1 x_4}\right)^2 \quad x \in \mathbb{Z}^4,\ 12 \le x_i \le 60$$

   (1.1) Is the problem a convex or nonconvex MINLP?

   (1.2) Is there an equivalent but simpler formulation?

2. The following model was published in an obscure paper, and concerns that optimization of IEEE 802.11 broadband networks for resource sharing meshes (Costa-Montenegro et al., 2007). The model contains the following nonlinear penalty that is added to the objective (with a large penalty parameter to ensure $z = 0$):

$$z = \frac{1}{1 + 1000(x-y)^{10}}$$

   (2.1) Plot this function (as a function of the difference $d = x - y$, e.g. using Matlab. What do you observe?

   (2.2) Can you deduce the modelers intention from the plot?

   (2.3) Model this expression as MIP not NLP!

3. Show that no integer assignment can be generated twice by outer approximation. Hint: consider the linearized constraints and the upper bound (or infeasibilty). This essentially proves convergence of outer approximation (correctness follows from convexity).

4. Consider the quadratic facility location problem from Lecture 1.

   (4.1) Formulate the model in AMPL. Construct the instances by placing facilities and locations randomly in the unit square and making the constant of proportionality for the transportation cost between facility $i \in I$ and customer $j \in J$ a function of the distance between the two locations. Specifically, if we (randomly) define the coordinates of the facilities and customers as

$$(x_1^F, y_1^F), \ldots, (x_M^F, y_M^F) \quad \text{and} \quad (x_1^C, y_1^C), \ldots, (x_N^C, y_N^C)$$

   respectively, then define the constant for pairwise service as

$$Q_{ij} = 50\sqrt{(x_i^F - x_j^C)^2 + (y_i^F - y_j^C)^2}$$

   and fix the cost as a uniform random variable between 0 and 100, which you can do in AMPL using `let{i in I} c[i] := 100*uniform(0,1).`

   (4.2) Experiment with different solvers. What do you observe?

   (4.3) Code outer approximation in `AMPL` for the quadratic facility location problem.

   Hints:

   - You only need to linearize the quadratic/logarithmic terms (add this to the `*.mod` file.
   - Introduce an objective variable, η, e.g, and parameters for lower/upper bounds, `LBD, UBD`.

- Define two optimization problems: master & subproblem using AMPL's `problem` statement. You can switch between these models by using `problem` again.
- Start at $z_i = 1, \ \forall i$ for simplicity.
- E.g. use `repeat { ... } until (Iter>=MaxIter || LBD>=UBD-1E-4);` for the loop.
- Fix integer variables in Subproblem by not including them in the `problem` statement!
- Add cuts to the master using by increasing a cut counter, and saving the cuts into a set of constraints.

How many iterations does OA take for the linear objective?

(4.4) Solve the QFL with disaggregated constraints, and conic reformulations (using the `gurobi` or `xpress` solver).

5. Consider the following nonconvex NLP from Lecture 2:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) := x_1^2 x_2^2 - 2x_1 x_2^3 + x_2^4 \\ \text{subject to} & -1 \leq x_i \leq 1 \ \text{ for } \ i = 1, 2 \end{array}$$

with three nonconvex terms and use the RLT ideas to obtain a lower bound (relaxation) in AMPL.

(5.1) What is the value of your lower bound, and how does it compare to the root relaxation of Baron?

(5.2) What solutions do the NLP solvers find (snopt, knitro, ipopt, filter, minos)? Perform a parametric search over $[-1, 1]^2$ and plot the value of the minimizer for different solvers.

(5.3) Can you obtain a tighter (and smaller) relaxation? **Hint:** Think about group partial separability!

(5.4) What is the tightest root-node relaxation bound that you can find?

6. Consider the following PDE-constrained optimization (PDECO) problem, see (OPTPDE, 2014; Tröltzsch, 1984), which describes a simple Poisson problem with distributed controls to reach a certain desired target state, u_d. We will spice up this problem with integer variables to make it more interesting:

$$\left\{ \begin{array}{ll} \underset{u,w}{\text{minimize}} & \frac{1}{2} \|u - u_d\|^2_{L^2(\Omega)} + 12 \int_\Gamma e_\Gamma \, u \, ds + \frac{1}{2} \|w\|_{L^x} \\ \text{subject to} & -\Delta u + u = w \ \text{ in } \Omega \\ & \frac{\partial u}{\partial n} = 0 \text{ on boundary } \Gamma \\ & w(x, y) \in \{0, 1\} \end{array} \right.$$

where $\Omega = [0, 1]^2$,

$$u_d = -\frac{142}{3} + 12 \left((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \right).$$

This exercise will try to determine which norm, e.g. L^1 or L^2, we should use for the regularization term of the binary control $w(x, y) \in \{0, 1\}$. Note, that for binary variables, the L^1-norm is the same as the L^2-norm, because $w^2(x, y) = |w(x, y)| = w(x, y)$ for $w(x, y) \in \{0, 1\}$.

(6.1) Write an AMPL model for this problem, using a 5-point finite-difference stencil, e.g.

```
subject to DiscretizePDE{(i,j) in IxI}:
   4*u[i,j] - u[i-1,j] - u[i+1,j] - u[i,j+1] - u[i,j-1]
      = ( w[i,j] - u[i,j] ) * (h^2) ;
```

where `I := 0..N` describes the discretization of the domain $\Omega = [0,1]^2$.

(6.2) Try the MINLP solvers `baron` and `knitro` on this problem. How does the solution time for the L^2 norm compare to the time for the L^1 norm? Do `gurobi` or `xpress` do any better?

(6.3) How can you explain this difference? Hint: "plot" the values of $w(x,y)$ of the relaxation (which you can obtain using, e.g., `snopt` or `knitro`.

References

Costa-Montenegro, E., González-Castaño, F. J., Rodriguez-Hernández, P. S., and Burguillo-Rial, J. C. (2007). Nonlinear optimization of IEEE 802.11 mesh networks. In *ICCS 2007, Part IV*, pages 466–473, Springer Verlag, Berlin.

OPTPDE (2014). OPTPDE — a collection of problems in PDE-constrained optimization. `http://www.optpde.net`.

Tröltzsch, F. (1984). The generalized bang-bang-principle and the numerical solution of a parabolic boundary-control problem with constraints on the control and the state. *Zeitschrift für Angewandte Mathematik und Mechanik*, 64(12):551–556.