Tutorial 5: Optimality Conditions

1. Consider the NLP

$$\underset{x}{\text{minimize}} \ \frac{1}{2}\left((x_1 - 1)^2 + x_2^2\right) \quad \text{subject to } -x_1 + \beta x_2^2 = 0.$$

   Write down its KKT and second-order conditions. For what values of β is $x^* = 0$ a local minimizer of this problem?

2. Write down the KKT conditions

$$\underset{l \le x \le u}{\text{minimize}} \ f(x)$$

   and establish the correctness of Theorem 7.1.1 by showing that the Lagrange multipliers are the slopes of $f(x)$ in the coordinate directions. *Hint:* Introduce "copy variables", and split $l \le x \le u$ into $x_l = x_u$, $x_l \ge l$, $-x_u \ge -u$, so that Theorem 8.2.1 can be applied.

3. Show that the KKT conditions of the LP around x^* (8.11) are equivalent to the KKT conditions of the NLP (8.2) at the solution x^*.

Tutorial 5: Optimality Conditions

1. Show that the conditions in Theorem 6.2.2 (optimality of TR subproblem) follow from the KKT conditions.
2. Consider the following optimization problem

$$\underset{x}{\text{minimize}}\ x^T A x \quad \text{subject to } \|x\|_2^2 = 1,$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix. Derive its optimality conditions, and show that it can be solved to *global* optimality. Does the same hold, if we maximize instead?
If you like, you can start using the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & -4 \end{bmatrix}$$

What is the interpretation of the Lagrange multiplier and the optimum, x^*?

Tutorial 5: Convexity and Duality

1. Show that the following functions are convex:
   - Linear function: $l(x) = a^T x + b$
   - Quadratic function: $q(x) = \frac{1}{2}x^T Gx + g^T x + b$ for pos. semi-def. Hessian, $G \succeq 0$
   - Norms, such as the ℓ_1 and ℓ_2 norm
   - Convex combinations of two convex functions, i.e. if $g(x), h(x)$ are convex, then $f(x) = (1-\lambda)g(x) + \lambda h(x)$ is convex for $\lambda \in [0, 1]$.
2. Prove the following theorem:

Theorem (KKT Conditions are Necessary and Sufficient)

KKT conditions are necessary and sufficient for a global minimum of a convex program.

3. Derive the dual of

$$\underset{x}{\text{minimize}} \;\; c^T x \quad \text{subject to } A^T x \geq b, \; x \geq 0$$