Modeling Optimization Problems: Case Study

GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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Outline

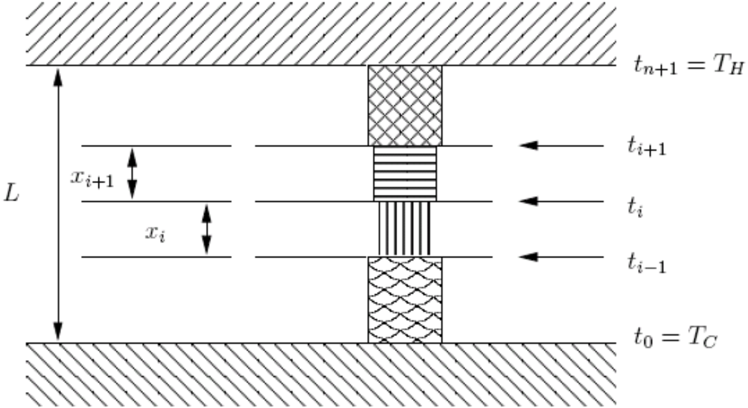
Modeling Optimization Problems: Case Study

Goal of our case study

- Work through concrete optimization example
  - Show interplay between variables & constraints
  - Build model from physics to math
    ... or from words to equations
  - Real physical design problem
- Highlight some modeling tricks
  ... preview of mixed-integer optimization part
- Continue our exploration of AMPL
- Show how good modeling improves solvability

Original model by Abramson (2004) modified by Abhishek et al. (2008)

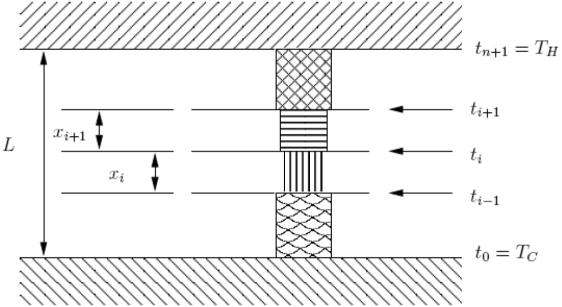
Design of Load-Bearing Thermal Insulation System



Description of System

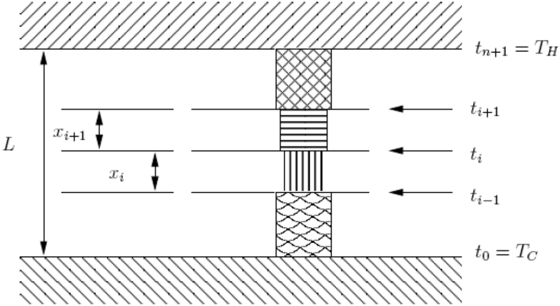
Insulation system uses series of heat intercepts
to reduce heat from hot (top) to cold (bottom) surface

Design of Load-Bearing Thermal Insulation System



- Similar to system used in Large Hadron Collider (LHC)
- Must be good insulator and support weight
- Uses insulation properties and cooling between layers (intercepts)
- Choose thickness of layers, and material type of layer
- Also optimize number of layers

Design of Load-Bearing Thermal Insulation System



Design Goal or Objective

Minimize cooling power needed to run system

- Active cooling at intercepts between layers $\Rightarrow$ cooling power
- Given hot surface temperature, maintain cold surface temperature below allowable maximum

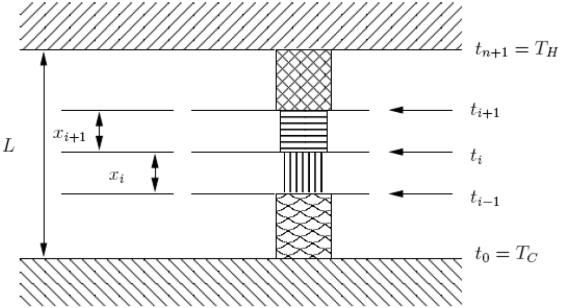
1D layers $\Rightarrow$ no heat equation: $u_t - \Delta u = g$ ... effects small

Outline

Discrete Design Variables Overview

Discrete design variables over which we optimize

- Number of intercepts, $n \in \{1, 2, \ldots, N = 10\}$ discrete
- m_i material $m_i \in \mathcal{M}$ of insulator $i = 1, \ldots, n + 1$
  where $m_i \in \mathcal{M} = \{$ nylon, teflon, epoxy-normal, epoxy-plane, aluminium, steel, carbon-steel $\}$ ... discrete choice



Continuous Design Variables Overview

Continuous variables over which we optimize

- x_i length of insulator $i = 1, \ldots, n+1$
- a_i area of insulator $i = 1, \ldots, n+1$
- q_i heat flow from intercept i to $i-1$, for $i = 1, \ldots, n+1$
- t_i cooling temperature at intercept $i = 0, \ldots, n+1$
- Δx_i thermal expansion of layer $i = 1, \ldots, n+1$
  can be eliminated later

where layers 0 and $n+1$ are cold and hot surface, respectively

- Cold surface temperature is $t_0 = T_C = 4.3$K (near abs. zero)
- Hot surface temperature is $t_n = T_H = 300$K (27C)

Objective Function

Minimize cooling power (discontinuous ... reformulate later)

$$\text{minimize} \quad \sum_{i=1}^{n} C(t_i) \left(\frac{T_H}{t_i} - 1 \right) \cdot (q_{i+1} - q_i)$$

where

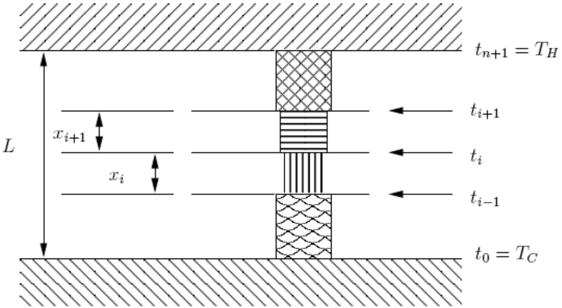
- $C(t_i)$ thermodynamic cycle efficiency of intercept i

$$C(t_i) = \begin{cases} 5 & \text{if } t_i \leq 4.2, \\ 4 & \text{if } 4.2 < t_i < 71 \,, \\ 2.5 & \text{if } t_i \geq 71. \end{cases} \quad i = 1, \ldots, n$$

- t_i is cooling temperature at intercept i
- q_i is heat flow from intercept i
- T_H=27C is ambient temperature $\Rightarrow \frac{T_H}{t_i} \geq 1$

Simple Linear Constraints

- $\sum_{i=1}^{n} x_i = L$ insulator thickness add up to L, length
- $t_{i-1} \leq t_i \leq t_{i+1},\ i = 1, \ldots, n$ ordered cooling temperatures
- $t_0 = T_C = 4.3\text{K}$ & $t_{n+1} = T_H = 300\text{K}$ fixed cold & hot temps
- $1 \leq n \leq N$ integer number of layers
- $x_i \geq 0,$ & $a_i \geq 0,\ i = 1, \ldots, n+1$ nonnegative thickness & area



Modeling Heat Transfer q_i

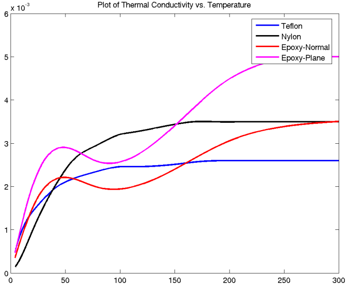
Given t_{i-1}, t_i, heat transfer from Fourier's law

$$q_i = \frac{a_i}{x_i} \int_{t_{i-1}}^{t_i} k(t, m_i) dt$$

where a_i area, x_i thickness of intercept i

Model of thermal conductivity

- $k(t, m_i)$ thermal conductivity of insulator m_i at temperature t
- $k(t, m_i)$ given as tabulated data
- interpolate using cubic splines
- integration with Simpson's rule
  $\Rightarrow$ consistent with cubic splines



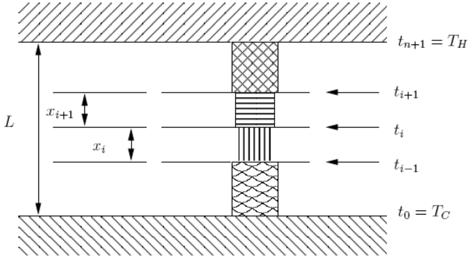
Modeling the Mass Constraint

Constraint on total mass of system:

$$\sum_{i=1}^{n} \rho(m_i) a_i x_i \leq M$$

where

- a_i area, x_i thickness of intercept i
- $\rho(m_i)$ is density of material m_i



Stress Limit Constraint

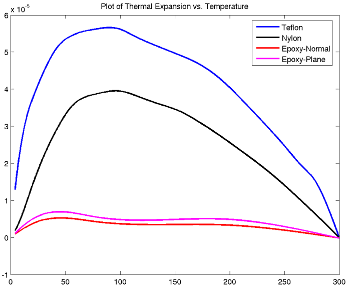
Stress of insulator i must not exceed load F

$$\frac{F}{a_i} \leq \bar{\sigma}_i = \min\left\{\sigma(t, m_i) \;:\; t_{i-1} \leq t \leq t_i\right\}$$

where

- $\sigma(t, m_i)$ tensile yield strength of insulator m_i at temp. t
- $\sigma(t, m_i)$ given as tabulated data
- interpolated using cubic splines
- write as semi-infinite constraint

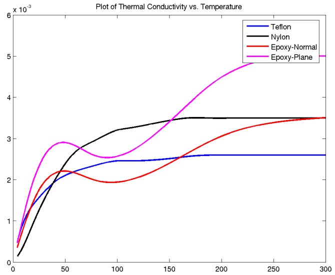
$$\Leftrightarrow \frac{F}{a_i} \leq \sigma(t, m_i) \;\forall\; t : \; t_{i-1} \leq t \leq t_i$$

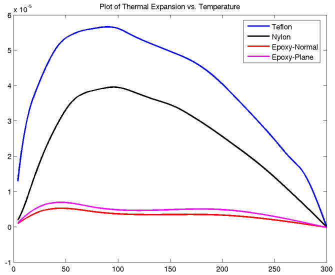


Teaching Point: Modeling Discrete Decisions

Materials { nylon, teflon, epoxy-normal, epoxy-plane, aluminium, steel, carbon-steel } are not linearly ordered

$\Rightarrow$ cannot assign numbers $1, \ldots 7$





Thermal Conductivity

Yield Strength

Modeling Thermal expansion of layers

Layers cannot expand more than δ %:

$$\sum_{i=1}^{n} \left(\frac{\Delta x_i}{x_i}\right)\left(\frac{x_i}{L}\right) \leq \frac{\delta}{100} \quad \Leftrightarrow \quad \sum_{i=1}^{n} u_i x_i \leq L\frac{\delta}{100}$$

where u_i relative expansion, replaces Δx_i.

Physical model of thermal expansion

$$\frac{\Delta x_i}{x_i} = u_i = \frac{\int_{t_{i-1}}^{t_i} e(t, m_i)k(t, m_i)dt}{\int_{t_{i-1}}^{t_i} k(t, m_i)dt}, \quad i = 1, \ldots, n$$

from thermal conductivity, $k(t, m_i)$, and yield strength, $e(t, m_i)$

Why did we introduce u_i?

Teaching Point: Spread the Nonlinearities

Why did we introduce u_i? Consider new variables, r, v, w

$$\left(\frac{r}{v}\right)\left(\frac{v}{L}\right) \leq D \quad \text{and} \quad \left(\frac{r}{v}\right) = \frac{f(r,v,w)}{g(r,v,w)}$$

where $f(r,v,w)$, $g(r,v,w)$ nonlinear functions, L, D constants

Clearly, can simplify 1^{st} constraint to $r \leq LD$

- Makes 1^{st} constraint linear
- Keeps $\frac{r}{v}$ in 2^{nd} constraint

By introducing $u = \frac{r}{v}$, we "spread the nonlinearity"

$$uv \leq LD \quad \text{and} \quad u = \frac{f(r,v,w)}{g(r,v,w)}$$

Best formulation depends on model ... no general rules!

Complete Mixed Variable Model

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} C_i(t_i)\left(\frac{T_H}{t_i}-1\right)\cdot(q_{i+1}-q_i) && \text{cooling power}\\
\text{subject to} \quad & q_i = \frac{a_i}{x_i}\int_{t_{i-1}}^{t_i} k(t,m_i)dt && \text{heat transfer}\\
& \sum_{i=1}^{n} \rho(m_i)a_i x_i \le M && \text{total mass}\\
& F \le a_i\sigma(t,m_i)\ \forall t:\ t_{i-1}\le t\le t_i && \text{stress limit}\\
& \sum_{i=1}^{n} u_i x_i \le L\frac{\delta}{100} && \text{thermal expansion}
\end{aligned}
$$

Plus linear constraints: $t_{i-1} \le t_i \le t_{i+1}, \quad x_i \ge 0, \quad a_i \ge 0$

$$
\sum_{i=1}^{n} x_i = L, \quad t_0 = T_C,\ t_{n+1} = T_H, \quad n \in \{1,\ldots,N\}, \quad m_i \in \mathcal{M}
$$

Outline

Exam: Code this Model in AMPL

Great model, let's see what the AMPL code looks like ...

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Great model, let's see what the AMPL code looks like ...

Are you kidding me??? This won't work in AMPL!!!

- How do I encode the material choice, which is non-integer?

  $$m_i \in \mathcal{M} = \{\text{nylon, teflon, epoxy-normal, ... steel, carbon-steel}\}$$

- How can I have variables as limits in summations?

  $$\sum_{i=1}^{n} x_i = L, \quad n \in \{1, \ldots, N = 10\}$$

  Would be written in AMPL as

```
param N := 10;              # ... max. number of layers
param L := 10;              # ... system length [cm]
var n, integer, >=1, <=N;   # ... number of layers
var x{1..n} >= 0;           # ... length of layer
subject to
   Length: sum{i in 1..n} x[i] = L;
```

Creates at least two ERRORs ... where?

Exam: Code this Model in AMPL

How can I have variables as limits in summations?

$$\sum_{i=1}^{n} x_i = L, \qquad n \in \{1, \ldots, N = 10\}$$

Would be written in AMPL as

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var n, integer, >=1, <=N; # ... number of layers
var x{1..n} >= 0;             # ... length of layer
subject to
   Length: sum{i in 1..n} x[i] = L;
```

Creates at least two ERRORs ... where?

- Variable n cannot be set limiter

  ```
  var x{1..n} >= 0;           # ... length of layer
  ```

- Variable n cannot be summation limiter

  ```
     Length: sum{i in 1..n} x[i] = L;
  ```

Other Issues with Our Model

- Even if AMPL allowed variable n as summation limit
  ... problem is discontinuous ... `sum` only defined for integers
- Objective function (cooling power) is discontinuous

$$\text{minimize} \sum_{i=1}^{n} C_i(t_i) \left(\frac{T_H}{t_i} - 1 \right) \cdot (q_{i+1} - q_i)$$

because

$$C(t_i) = \begin{cases} 5 & \text{if } t_i \leq 4.2, \\ 4 & \text{if } 4.2 < t_i < 71 \\ 2.5 & \text{if } t_i \geq 71. \end{cases}, \quad i = 1, \ldots, n$$

... most AMPL solvers require smooth functions

Other Issues with Our Model

Evaluation of heat transfer integrals requires quadrature

$$q_i = \frac{a_i}{x_i} \int_{t_{i-1}}^{t_i} k(t, m_i) dt$$

- Can in principle "code" quadrature rules in AMPL
- Resulting would be highly nonlinear,
  because integrals depend on variable t_i
- AMPL is an interpretive not compiled language
  $\Rightarrow$ quadrature rules would make function evaluation inefficient

Towards an AMPL Model

How do we solve this pesky problem?

- Code functions in `Fortran` or `C/C++`
  $\Rightarrow$ efficient function evaluations
- No derivatives, consider derivative-free solvers, e.g. pattern-search, simulated annealing, or evolutionary algorithms

Snag: Solver looses model insight ... black-box optimization

Can we still use AMPL to solve this model?

- Requires some modeling tricks
- Resulting model is a smooth mixed-integer nonlinear program
- Efficient ... optimum 10% better than derivative-free solvers

Conclusion and Outlook

Introduced a challenging case study

- Design of thermal insulation layer for LHC
  - Goal is to minimize cooling energy
  - Constraints on thermal insulation, mass, stress limits
  - Discrete design choices: number of layers, material
  - Continuous design parameters: thickness, area, temperature
- Translated physical description into mathematical model
- Numerically challenging problem ... no obvious AMPL model
- Integer modeling tricks (next week) make it tractable

Return to model later, when we learn about integer variables