Tutorial 4: Newton Methods

- Show that Newton's method oscillates for $\min\ f(x) = x^2 - x^4/4$. Starting from $x^{(0)} = \sqrt{2/5}$ it creates alternating iterates $-\sqrt{2/5}$ and $\sqrt{2/5}$.
- Show that the quasi-Newton condition, $B\gamma = \delta$ holds for a quadratic function.
- Apply Newton's method to nonlinear least-squares:

$$\underset{x}{\text{minimize}}\ f(x) = \sum_{i=1}^{m} r_i(x)^2 = r(x)^T r(x) = \|r(x)\|_2^2.$$

  What happens, if $r_i(x)$ are linear? Can you propose a strategy for handling the case, where $\nabla^2 r_i(x)$ are bounded, and $r_i(x) \to 0$?
  This is the basis of the Gauss-Newton method.

Tutorial 4: Stationarity

- Consider

$$f(x) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1).$$

  Find its gradient and Hessian matrix, and find and classify all its stationary points.
  If you like, plot $f(x)$ in domain $[-1,1]^2$ using Matlab:

```
xx = [-1:0.05:1];
[x,y] = meshgrid(xx,xx);
f = 2*x.^3 - 3*x.^2 - 6*x.*y.*(x - y - 1);
surfc(x,y,f);
```

  Use Matlab's `help` function to understand this code!
  Why is there a "." before the $\hat{}$?

- Repeat the previous question for Powell's function:

$$f(x) = x_1^4 + x_1x_2 + (1 + x_2)^2$$

Tutorial 4: Bound Constrained Optimization

- For quadratic, $q(x) = g^T x + \frac{1}{2} x^T G x$, give an explicit formula for the Cauchy Point in (6.7).
  - Given $\hat{x}$ find the steepest descend direction $\hat{s} = -\nabla q(\hat{x})$
  - Minimize the quadratic from $\hat{x}$ in the direction $\hat{s}$:

$$\underset{\alpha}{\text{minimize}} \;\; q(\hat{x} + \alpha \hat{s}) \quad \text{subject to } \|\alpha \hat{s}\| \leq \Delta$$

- Solve the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; c^T x \quad \text{subject to } l \leq x \leq u,$$

  where $c, l, u \in \mathbb{R}^n$ and $-\infty < l \leq u < \infty$.
  What happens if some l_i or u_i are not finite?
- Show that the two version of bound constrained optimality are equivalent, i.e.

$$x^* = P_{[l,u]}\left(x^* - \nabla f(x^*)\right) \;\Leftrightarrow\; \frac{\partial f}{\partial x_i}(x^*) \begin{cases} \geq 0, & \text{if } x_i^* = l_i \\ = 0, & \text{if } l_i < x_i^* < u_i \\ \leq 0, & \text{if } x_i^* = u_i. \end{cases}$$