# Mixed-Integer Nonlinear Optimization: Cutting Planes

## GIAN Short Course on Optimization: Applications, Algorithms, and Computation

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# Outline

1. Branch-and-Cut for MINLP
2. Cutting Planes for MINLP
    - Perspective Cuts
    - Disjunctive Cuts

# Mixed-Integer Nonlinear Optimization

Mixed-Integer Nonlinear Program (MINLP)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c(x) \leq 0 \\ & x \in X \\ & x_i \in \mathbb{Z} \text{ for all } i \in I \end{aligned}$$

Assumptions:

- A1 $X$ is a bounded polyhedral set.
- A2 $f$ and $c$ are twice continuously differentiable convex functions.
- A3 MINLP satisfies a constraint qualification.

Look at another class of branch-and-cut methods ...

# Overview of Branch-and-Cut Methods

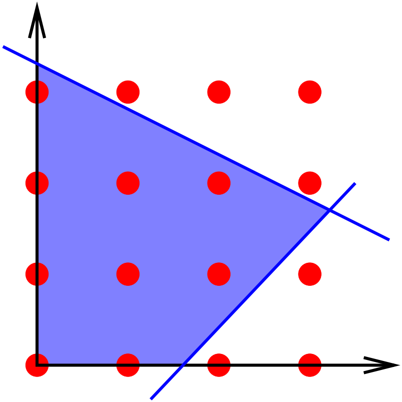
Extend nonlinear branch-and-bound

1. Solve NLP($l, u$) at each node of tree
   - Generate a cut to eliminate fractional solution & re-solve
   - Only branch if solution fractional after some rounds of cuts
2. Generation of good cuts is key [Stubbs and Mehrotra, 1999]
3. Hope that tree is smaller than BnB
4. Goal: get formulation closer to convex hull

# Motivation for Branch-and-Cut Methods

Consider Polyhedron over Integer Lattice

$\Rightarrow$ minimizing $c^T x$ over set, gives (fractional) vertex



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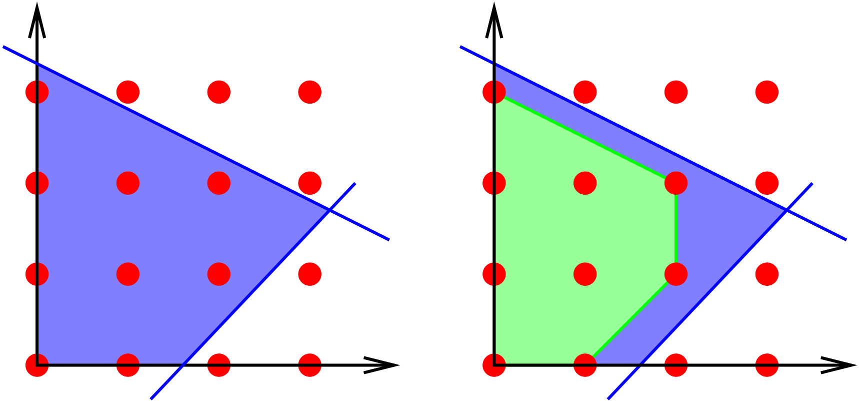


Consider Convex Hull of Feasible Integers

$\Rightarrow$ minimizing $c^T x$ over set, gives integer vertex

# Recall Nonlinear Branch-and-Bound

Solve NLP relaxation

$$\underset{x}{\text{minimize}} \;\; f(x) \quad \text{subject to } c(x) \leq 0,\; x \in X$$

- If $x_i \in \mathbb{Z} \; \forall \; i \in I$, then solved MINLP
- If relaxation is infeasible, then MINLP infeasible

... otherwise search tree whose nodes are NLPs:

$$\begin{cases} \underset{x}{\text{minimize}} & f(x), \\ \text{subject to} & c(x) \leq 0, \\ & x \in X, \\ & l_i \leq x_i \leq u_i, \; \forall i \in I. \end{cases} \qquad (\text{NLP}(l,u))$$

NLP relaxation is $\text{NLP}(-\infty, \infty)$

# Branch-and-Bound for MINLP

**Branch-and-bound for MINLP**

Choose tol $\epsilon > 0$, set $U = \infty$, add (NLP($-\infty, \infty$)) to heap $\mathcal{H}$.

**while** $\mathcal{H} \neq \emptyset$ **do**

Remove (NLP($l, u$)) from heap: $\mathcal{H} = \mathcal{H} - \{ \text{NLP}(l, u) \}$.

Solve (NLP($l, u$)) $\Rightarrow$ solution $x^{(l,u)}$

**if** *(NLP($l, u$)) is infeasible* **then**

Prune node: infeasible

**else if** $f(x^{(l,u)}) > U$ **then**

Prune node; dominated by bound $U$

**else if** $x_I^{(l,u)}$ *integral* **then**

Update solution: $U = f(x^{(l,u)})$, $x^* = x^{(l,u)}$.

**else**

BranchOnVariable($x_i^{(l,u)}, l, u, \mathcal{H}$)

**end**

**end**

# Generic Nonlinear Branch-and-Cut

Choose a tol $\epsilon > 0$, set $U = \infty$, add (NLP$(-\infty, \infty)$) to heap $\mathcal{H}$.

**while** $\mathcal{H} \neq \emptyset$ **do**

Remove (NLP$(l, u)$) from heap: $\mathcal{H} = \mathcal{H} - \{$ NLP$(l, u)$ $\}$.

**repeat**

Solve (NLP$(l, u)$) $\Rightarrow$ solution $x^{(l,u)}$.

**if** *(NLP$(l, u)$) is infeasible* **then**

Prune node: infeasible

**else if** $f(x^{(l,u)}) > U$ **then**

Prune node; dominated by bound $U$

**else if** $x_I^{(l,u)}$ *integral* **then**

Update incumbent: $U = f(x^{(l,u)})$, $x^* = x^{(l,u)}$ & prune.

**else** GenerateCuts$(x^{(l,u)}, j)$ ... details later ;

**until** *no new cuts generated or node pruned*;

**if** *(NLP$(l, u)$) not pruned & not incumbent* **then**

BranchOnVariable$(x_j^{(l,u)}, l, u, \mathcal{H})$

**end**

**end**

# Cut Generation Overview

**Subroutine: GenerateCuts** $(x^{(l,u)}, j)$

*// Generate a valid inequality that cuts off* $x_j^{(l,u)} \notin \{0,1\}$

Solve separation (NLP) problem in $x^{(l,u)}$ for valid cut.

Add valid inequality to (NLP$(l,u)$).

**GenerateCuts**: valid inequality to eliminate fractional solution

- Given fractional solution $x^{(l,u)}$ with $x_j^{(l,u)} \notin \{0,1\}$.
- Let $\mathcal{F}(l,u)$ mixed-integer feasible set of node NLP$(l,u)$.
- Find cut $\pi^T x \leq \pi_0$ such that
  - $\pi^T x \leq \pi_0$ for all $x \in \mathcal{F}(l,u)$
  - $\pi^T x^{(l,u)} > \pi_0$, i.e. $x^{(l,u)}$ violates the cut
- Solve a separation problem (e.g. an NLP) for cut $\pi^T x \leq \pi_0$

... lifting cuts makes them valid throughout the tree.

# Example: Mixed-Integer Rounding (MIR) for MILP

Goal: Strengthen MILP relaxations of LP/NLP-based BnB
... iteratively add cuts to remove fractional LP solutions

Start by considering MIR cuts for "easy set"

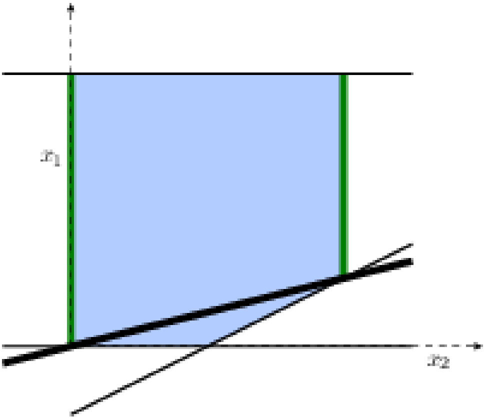
$$S := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{Z} \mid x_2 \leq b + x_1,\ x_1 \geq 0\},$$

Let $f_0 = b - \lfloor b \rfloor$, then cut can show that

$$x_2 \leq \lfloor b \rfloor + \frac{x_1}{1 - f_0}$$

is valid for $S$

# Example of Simple “MIR” Cut




MIR cut: $x_2 \leq 2x_1$ derived from $x_2 \leq \frac{1}{2} + x_1$.

Closer to convex hull $\Rightarrow$ integral solutions to relaxation

# Branch-and-Cut Challenges

Computational Considerations of Branch-and-Cut

- Cut-generation problem may be hard to solve
- Adds burden of additional NLP solves to BnB
    - Can solve LP instead of NLP, e.g. from OA
- Must add cut-management to solver
- Lifting cuts may help to make them valid in whole tree
- NLPs still don't hot-start

[Stubbs and Mehrotra, 1999] generate cuts only at root node

# Outline

# Perspective Formulations

MINLPs use binary indicator variables, $x_b$, to model nonpositivity of $x_c \in \mathbb{R}$

Model as variable upper bound

$$0 \leq x_c \leq u_c x_b, \quad x_b \in \{0, 1\}$$

$\Rightarrow$ if $x_c > 0$, then $x_b = 1$

Perspective reformulation applies, if $x_b$ also in convex $c(x) \leq 0$

- Significantly improve reformulation
- Pioneered by [Frangioni and Gentile, 2006];
  ... strengthen relaxation using perspective cuts

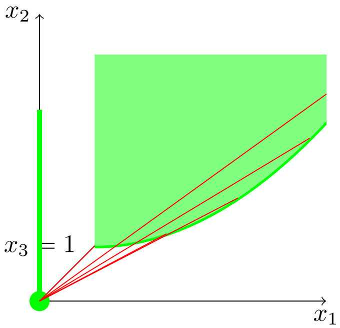
# Example of Perspective Formulation

Consider MINLP set with three variables:

$$S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^2 \times \{0,1\} \ : \ x_2 \geq x_1^2, \quad ux_3 \geq x_1 \geq 0 \right\}.$$

Can show that $S = S^0 \cup S^1$, where

$$S^0 = \left\{ (0, x_2, 0) \in \mathbb{R}^3 \ : \ x_2 \geq 0 \right\},$$
$$S^1 = \left\{ (x_1, x_2, 1) \in \mathbb{R}^3 \ : \ x_2 \geq x_1^2, \ u \geq x_1 \geq 0 \right\}.$$



# Example of Perspective Formulation

Geometry of convex hull of $S$:
Lines connecting origin ($x_3 = 0$) to parabola $x_2 = x_1^2$ at $x_3 = 1$

Define convex hull of $S$ as conv($S$)

$$:= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 x_3 \geq x_1^2,\ u x_3 \geq x_1 \geq 0, 1 \geq x_3 \geq 0, x_2 \geq 0\right\}$$

where $x_2 x_3 \geq x_1^2$is defined in terms of perspective function

$$\mathcal{P}_f(x,z) := \begin{cases} 0 & \text{if } z = 0, \\ zf(x/z) & \text{if } z > 0. \end{cases}$$

Epigraph of $\mathcal{P}_f(x,z)$: cone pointed at origin with lower shape $f(x)$

$x_b \in \{0,1\}$ indicator forces $x_c = 0$, or $c(x_c) \leq 0$ if $x_b = 1$ write

$$x_b c(x_c / x_b) \quad \text{...is tighter convex formulation}$$

# Generalization of Perspective Cuts

[Günlük and Linderoth, 2012] consider more general problem

$$(P) \qquad \min_{(x,z,\eta)\in\mathbb{R}^n\times\{0,1\}\times\mathbb{R}} \Big\{\eta \mid \eta \geq f(x) + cz, Ax \leq bz\Big\}.$$

where

1. $X = \{x \mid Ax \leq b\}$ is bounded
2. $f(x)$ is convex and finite on $X$, and $f(0) = 0$

**Theorem (Perspective Cut)**

*For any $\bar{x} \in X$ and subgradient $s \in \partial f(\bar{x})$, the inequality*

$$\eta \geq f(\bar{x}) + c + s^T(x - \bar{x}) + (c + f(\bar{x}) - s^T\bar{x}))(z - 1)$$

*is valid cut for (P)*

# Stronger Relaxations [Günlük and Linderoth, 2012]

- $z_R$: Value of NLP relaxation
- $z_{GLW}$: Value of NLP relaxation after GLW cuts
- $z_P$: Value of perspective relaxation
- $z^*$: Optimal solution value

Separable Quadratic Facility Location Problems

| $\|M\|$ | $\|N\|$ | $z_R$ | $z_{GLW}$ | $z_P$ | $z^*$ |
|---|---|---|---|---|---|
| 10 | 30 | 140.6 | 326.4 | 346.5 | 348.7 |
| 15 | 50 | 141.3 | 312.2 | 380.0 | 384.1 |
| 20 | 65 | 122.5 | 248.7 | 288.9 | 289.3 |
| 25 | 80 | 121.3 | 260.1 | 314.8 | 315.8 |
| 30 | 100 | 128.0 | 327.0 | 391.7 | 393.2 |

$\Rightarrow$ Tighter relaxation gives faster solves!

# Disjunctive Branch-and-Cut

[Stubbs and Mehrotra, 1999] for convex, binary MINLP:

$$\underset{\eta,x}{\text{minimize}}\ \eta \quad \text{s.t. } \eta \geq f(x),\ c(x) \leq 0,\ x \in X,\ x_i \in \{0,1\}\ \forall\ i \in I$$

Node in BnB tree with solution $x'$, and $0 < x'_j < 1$ for $j \in I$
Relaxation: $\mathcal{C} = \{x \in X \mid f(x) \leq \eta,\ c(x) \leq 0,\ 0 \leq x_I \leq 1\}$
Let $I_0, I_1 \subseteq I$ index sets of 0-1 vars fixed to zero or one

Goal: Generate a valid inequality tat cuts off $x'$
Consider two disjoint sets ("feasible sets after branching on $x_j$")

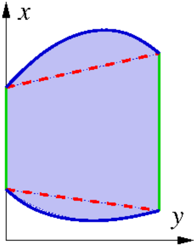
$$\mathcal{C}_j^0 = \{x \in \mathcal{C} \mid x_j = 0,\ 0 \leq x_i \leq 1\ \forall i \in I, i \neq j\},$$
$$\mathcal{C}_j^1 = \{x \in \mathcal{C} \mid x_j = 1,\ 0 \leq x_i \leq 1\ \forall i \in I, i \neq j\}.$$

... and find description of convex hull: $\tilde{M}_j(\mathcal{C}) = \text{conv}(\mathcal{C}_j^0 \cup \mathcal{C}_j^1)$

# Disjunctive Cuts for MINLP

Extension of disjunctive cuts from MILP, [Balas, 1979]
Continuous relaxation

- $\mathcal{C} := \{x | c(x) \leq 0,\ 0 \leq x_I \leq 1,\ 0 \leq x_C \leq U\}$



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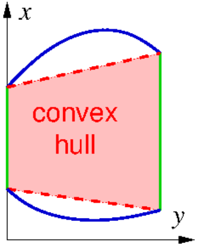
- $\mathcal{C} := \text{conv}(\{x \in \mathcal{C} \mid x_I \in \{0,1\}^p\})$



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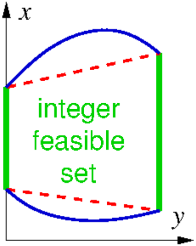
- $\mathcal{C} := \text{conv}(\{x \in \mathcal{C} \mid x_I \in \{0,1\}^p\})$
- $\mathcal{C}_j^{0/1} := \{x \in \mathcal{C} | x_j = 0/1\}$




$$\text{let } \mathcal{M}_j(C) := \left\{ \begin{array}{l} z = \lambda_0 u_0 + \lambda_1 u_1 \\ \lambda_0 + \lambda_1 = 1,\ \lambda_0, \lambda_1 \geq 0 \\ u_0 \in \mathcal{C}_j^0,\ u_1 \in \mathcal{C}_j^1 \end{array} \right\}$$

$\Rightarrow \mathcal{P}_j(\mathcal{C}) :=$ projection of $\mathcal{M}_j(\mathcal{C})$ onto $z$

$\Rightarrow \mathcal{P}_j(\mathcal{C}) = \text{conv}\left(\mathcal{C} \cap x_j \in \{0,1\}\right)$ and $\mathcal{P}_{1\ldots p}(\mathcal{C}) = \mathcal{C}$

# Disjunctive Cuts

Snag: Description of convex hull is nonconvex:

$$\text{let } \mathcal{M}_j(\mathcal{C}) := \left\{ \begin{array}{l} z = \lambda_0 u_0 + \lambda_1 u_1 \\ \lambda_0 + \lambda_1 = 1,\ \lambda_0, \lambda_1 \geq 0 \\ u_0 \in \mathcal{C}_j^0,\ u_1 \in \mathcal{C}_j^1 \end{array} \right\}$$

$\Rightarrow$ need global optimization solvers for separation problem

$\Rightarrow$ prohibitive; instead use convex formulation: $\tilde{M}_j(\mathcal{C})$

# Disjunctive Cuts

Can describe $\tilde{M}_j(\mathcal{C})$ with perspective $\mathcal{P}_{c_i}$

$$\tilde{M}_j(\mathcal{C}) = \left\{ (x_F, v_0, v_1, \lambda_0, \lambda_1) \;\middle|\; \begin{array}{ll} v_0 + v_1 = x_F, & v_{0j} = 0, \; v_{1j} = \lambda_1 \\ \lambda_0 + \lambda_1 = 1, & \lambda_0, \lambda_1 \geq 0 \\ \lambda_0 c_i(v_0/\lambda_0) \leq 0, 1 \leq \; i \leq m & \\ \lambda_1 c_i(v_1/\lambda_1) \leq 0, 1 \leq \; i \leq m & \end{array} \right\},$$

Obtain a convex separation NLP ...

# Disjunctive Cuts: Separation NLP

Goal: Find $\hat{x}$ closest to fractional solution $x'$ in convex hull

$$
\text{BC-SEP}(x',j) \begin{cases} \underset{x,v_0,v_1,\lambda_0,\lambda_1}{\text{minimize}} & ||x - x'||, \\ \text{subject to} & (x, v_0, v_1, \lambda_0, \lambda_1) \in \tilde{M}_j(\mathcal{C}) \\ & x_i = 0, \ \forall i \in I_0 \\ & x_i = 1, \ \forall i \in I_1. \end{cases}
$$

optimal solution $\hat{x}$ with multipliers $\pi_F$ for equality $v_0 + v_1 = x_F$

**Theorem**

*Optimal dual solution of (BC-SEP$(x',j)$), then following cut is valid and eliminates $x'$:*

$$\pi_F^T x_F \le \pi_F^T \hat{x}_F$$

# Disjunctive Cuts: Example

Consider following MINLP example

$$\begin{cases} \underset{x_1, x_2}{\text{minimize}} & x_1 \\ \text{subject to} & (x_1 - \frac{1}{2})^2 + (x_2 - \frac{3}{4})^2 \leq 1 \\ & -2 \leq x_1 \leq 2 \\ & x_2 \in \{0, 1\} \end{cases}$$

$\Rightarrow$ solution of NLP relaxation: $x' = (x'_1, x'_2) = (-\frac{1}{2}, \frac{3}{4})$

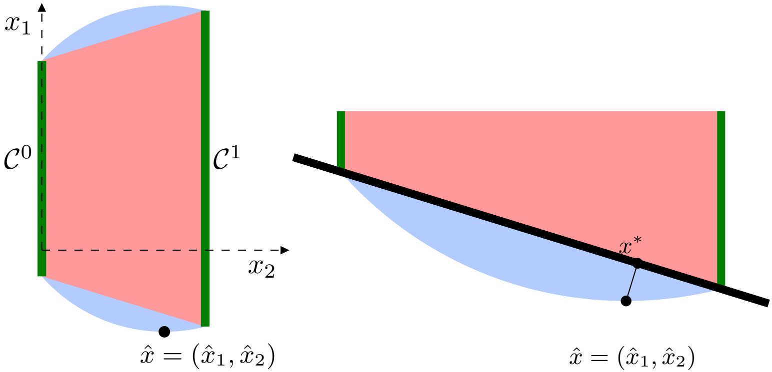
Solve $(x_1 - \frac{1}{2})^2 + (x_2 - \frac{3}{4})^2 \leq 1$ for $x_1$, given $x_2 = 0$ and $x_2 = 1$:

$$\mathcal{C}^0 = \left\{ (x_1, 0) \in \mathbb{R} \times \{0, 1\} \;\middle|\; 2 - \sqrt{7} \leq 4x_1 \leq 2 + \sqrt{7} \right\},$$

$$\mathcal{C}^1 = \left\{ (x_1, 1) \in \mathbb{R} \times \{0, 1\} \;\middle|\; 2 - \sqrt{15} \leq 4x_1 \leq 2 + \sqrt{15} \right\}.$$

Solving (BC-SEP$(x', 2)$), we find the cut $x_1 + 0.3x_2 \geq -0.166$

# Disjunctive Cuts: Example




Convex hull, relaxation, and disjunctive cut

# Lifting Disjunctive Cuts

Cuts are only valid for sub-tree rooted at relaxation
To obtain globally valid cut

$$\pi^T x \leq \pi^T \hat{x}$$

assign

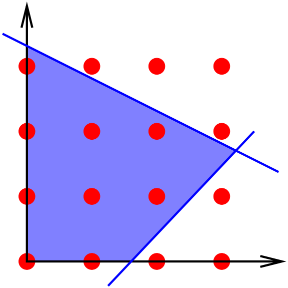
$$\pi_i = \min\{e_i^T H_0^T \mu_0, e_i^T H_1^T \mu_1\}, \; i \notin F$$

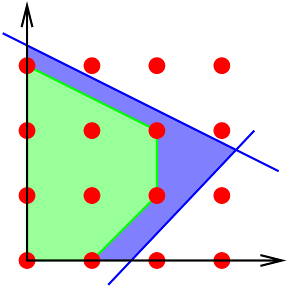
where $e_i$ is $i^{th}$ unit vector, $F$ set of "free" variables and

- $\mu_0 = (\mu_{0F}, 0)$ and $\mu_{0F}$ multiplier of perspective $\mathcal{P}_c(v_0, \lambda_0) \leq 0$
- $\mu_1 = (\mu_{1F}, 0)$ and $\mu_{1F}$ multiplier of perspective $\mathcal{P}_c(v_1, \lambda_1) \leq 0$
- $H_0$, $H_1$ matrices of subgradient rows $\partial_v \mathcal{P}_{c_i}(v_j, \lambda_j)^T$, for $j = 0, 1$

Preferred norm for cut generation, (BC-SEP$(x', j)$), is $\ell_\infty$-norm

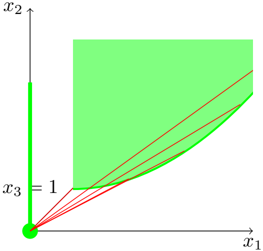
# Summary and Teaching Points

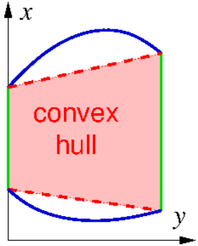




Classes of Cuts

1. Perspective cuts
2. Disjunctive cuts



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