Exercise: Solving the Trust-Region Subproblem

$$\underset{w \in \{0,1\}^p}{\text{minimize}} \ \mathcal{F}^{(k)} + g^{(k)^T} \left(w - w^{(k)} \right) \ \text{ subject to } \|w - w^{(k)}\|_1 \le \Delta_k,$$

Exercise: Solve TR-subproblem in $\mathcal{O}(N)$ steps; N dimension of w!

1. Re-write trust-region constraint as knapsack-like constraint:

$$\|w - w^{(k)}\|_1 \le \Delta_k \quad \Leftrightarrow \quad \sum_{w_i^{(k)}=0} w_i - \sum_{w_i^{(k)}=1} w_i \le \Delta_k - \sum_{w_i^{(k)}=1} 1$$

2. Distinguish four cases
   1. $w_i^{(k)} = 0$ and $g_i^{(k)} \ge 0$
   2. $w_i^{(k)} = 1$ and $g_i^{(k)} \le 0$
   3. $w_i^{(k)} = 0$ and $g_i^{(k)} < 0$
   4. $w_i^{(k)} = 1$ and $g_i^{(k)} > 0$

Construct/solve knapsack problem: $\underset{\hat{w}}{\min} \ \hat{g}^T \hat{w} \quad \text{s.t.} \ \mathbb{1}^T \hat{w} \le \delta$